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Physics Letters B 632 (2006) 422–426
PHYSICS LETTERS B
www.elsevier.com/locate/physletb

ζ-function regularization of the effective action for a δ-function potential

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Received 21 September 2005; accepted 14 October 2005

Available online 24 October 2005

Editor: N. Glover

Abstract

We present an exact evaluation of the generalized ζ-function for a real scalar field in a δ-function potential in the spacetime $\mathbb{S}^1 \times \mathbb{R}^{D-1}$. The result for the ζ-function is used to obtain the effective action. As a byproduct of the calculation the heat-kernel coefficients are obtained to all orders in a closed form. The regularized zero-point energy is found and we discuss the renormalization of the effective action.

PACS: 11.10.Gh; 11.10.-z; 03.70.+k

Keywords: ζ-function; Casimir effect; Zero-point energy; Renormalization; Heat-kernel coefficients

1. Introduction

The use of ζ-function regularization in quantum field theory dates back to Refs. [1,2]. It is closely related to the use of the heat-kernel expansion, whose utility in general quantum field theory was described by DeWitt [3]. There is also a large body of mathematical literature on generalized ζ-functions and the asymptotic expansion of the heat kernel. (See Ref. [4] for example.) Extensive references to the literature on ζ-function regularization and related matters, particularly quantum field theory applications, can be found in the reviews [5–7].

The main concern of the present Letter is the explicit evaluation of the generalized ζ-function for a real scalar field in a δ-function potential, and its use in obtaining a general expression for the heat-kernel coefficients to arbitrary order, in the spacetime $\mathbb{S}^1 \times \mathbb{R}^{D-1}$. Apart form interest simply as a possible model of an external potential, δ-function type potentials arise naturally in brane-world models [8]. (See Refs. [9–12] for a selection of the earliest papers on quantum field theory in brane-world models.) The heat-kernel asymptotic expansion in the presence of a δ-function potential has been studied in Refs. [13–15]. Early work on quantum field theory in the presence of δ-function potentials, especially the computation of vacuum energies was presented in Ref. [16] based on earlier evaluations of the Feynman Green function [17]. Later work on vacuum energies and other quantum field theory calculations includes [18–22].

We begin with the usual action functional for a scalar field ϕ with a coupling to an external potential $V(x)$ in D spacetime dimensions.

$$S_0 = \int d^D x \left\{ \frac{1}{2}\partial^\mu \phi \partial_\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} V(x) \phi^2 \right\}. \tag{1.1}$$

We will mention briefly what happens for a self-interaction at the end when we discuss the renormalization of the theory. The spacetime of interest to us here is $\mathbb{S}^1 \times \mathbb{R}^{D-1}$, and we choose to work with a Euclidean metric. To deal with the infinite volume we will assume periodic boundary conditions and take the large volume limit. We will choose the $\mathbb{S}^1$ direction to be x where $0 \leqslant x \leqslant L$, L representing the circumference of the circle that we keep finite. The periodicity lengths in the other spatial directions are denoted by $L_2, \dots, L_{D-1}$ and we take $L_j \gg L$ for all $j = 2, \dots, D-1$. The fields are also periodic in the time direction with period β, and since we will not be concerned with finite temperature effects here we take $\beta \gg L$ as well.

As an external potential we take

$$V(x) = V_0 \delta(x - a), \tag{1.2}$$

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doi:10.1016/j.physletb.2005.10.031

where $0 < a < L$, and $V_0 > 0$ is a constant. Due to symmetry under rotations around $\mathbb{S}^1$ we would not expect our result for the effective action, and hence for any observable consequence, to depend on the value chosen for a. We define

$$\Delta = -\Box + m^2 + V(x), \tag{1.3}$$

and the one-loop contribution to the effective action is

$$\Gamma^{(1)} = \frac{1}{2}\ln\det\bigl(\ell^2\Delta\bigr), \tag{1.4}$$

where ℓ is an arbitrary renormalization scale with dimensions of length. We will regularize this by ζ-function regularization [2]. To do this, let $\phi_k(x)$ be an eigenfunction of Δ with eigenvalue λ_k:

$$\Delta\phi_k = \lambda_k\phi_k. \tag{1.5}$$

Define

$$\zeta(s) = \sum_k \lambda_k^{-s} \tag{1.6}$$

for $\Re(s) > D/2$. By analytic continuation we can compute $\zeta(s)$ in a neighbourhood of $s = 0$ and adopt

$$\Gamma^{(1)} = -\frac{1}{2}\zeta'(0) + \frac{1}{2}\zeta(0)\ln\ell^2. \tag{1.7}$$

Knowledge of λ_k enables us to obtain $\zeta(s)$ which in turn allows us to evaluate $\Gamma^{(1)}$.

The generalized ζ-function is intimately connected to the heat kernel,

$$K(\tau) = \sum_k e^{-\tau\lambda_k}. \tag{1.8}$$

As $\tau \to 0$, $K(\tau)$ has a well-known asymptotic expansion that we will write as

$$K(\tau) \sim (4\pi\tau)^{-D/2}\sum_{j=0}^{\infty}\bigl(\tau^j a_j + \tau^{j+1/2}b_j\bigr) \tag{1.9}$$

for coefficients a_j and b_j. These heat-kernel coefficients are known under very general conditions. For the δ-function potential, the existence of this expansion has been shown in a more general setting by Bordag and Vassilevich [13]. The importance of the generalized ζ-function is that $(4\pi)^{D/2}\Gamma(s)\zeta(s)$ has simple poles at $s = D/2 - j$ with residues a_j and at $s = (D-1)/2 - j$ with residues b_j for $j = 0, 1, 2, \ldots$. Therefore, if we can compute $\zeta(s)$ and then study its analytic continuation to $\Re(s) < D/2$ we can calculate the heat-kernel coefficients.

For the case of δ-function potentials, the first few heat-kernel coefficients are known in a variety of cases [13–15]. What we will do in the present Letter is to calculate an explicit result for $\zeta(s)$ in the simple spacetime described above, obtain its analytic continuation and discover explicit results for a_j and b_j for all j. The results can be checked against previously known ones for small j, and will provide a useful consistency check on any future work in this area. In addition, we will find explicit closed form results for $\zeta(0)$ and $\zeta'(0)$ thereby calculating the effective action. This essentially determines the Casimir vacuum energy obtained by a regularized sum of zero-point energies.

2. Generalized ζ-function

It is straightforward to show that the eigenvalues λ_k in (1.5) for the particular case of (1.3) with $V(x)$ given by (1.2) is

$$\lambda_k = \left(\frac{2\pi n_0}{\beta}\right)^2 + \left(\frac{2\pi n_2}{L_2}\right)^2 + \cdots + \left(\frac{2\pi n_{D-1}}{L_{D-1}}\right)^2 + k_x^2 + m^2, \tag{2.1}$$

with periodic boundary conditions as discussed. $n_0, n_2, \ldots, n_{D-1}$ are integers taking the values $0, \pm1, \pm2, \ldots$ and k_x is the solution to

$$\psi(\kappa) = \sin(\kappa/2) - \alpha\frac{\cos(\kappa/2)}{\kappa} = 0, \tag{2.2}$$

where we have defined dimensionless variables

$$\alpha = \frac{1}{2}LV_0, \tag{2.3}$$

$$\kappa = k_x L. \tag{2.4}$$

For $\alpha = 0$, this gives the usual result $k_x = 2\pi n/L$ for periodic boundary conditions. For $\alpha > 0$, the solutions to (2.2) are all real and non-zero. We can label the solutions to (2.2) by κ_n with $n = \pm1, \pm2, \ldots$ and $\kappa_{-n} = -\kappa_n$. Of course for $\alpha \neq 0$ we cannot find an explicit result for κ_n, but we will see that we do not need to. The derivation of (2.2) is standard: Just take the general solutions for $x < a$ and $x > a$ and match the discontinuity in derivatives across the δ-function. The location of the δ-function (namely the value of a) does not enter the eigenvalues, as expected.

Form the generalized ζ-function (1.6) and replace the sums over $n_0, n_2, \ldots, n_{D-1}$ with integrals, valid for $\beta, L_2, \ldots, L_{D-1} \gg L$. These integrals are easily evaluated using the integral representation of the Γ-function, leaving

$$\zeta(s) = V(4\pi)^{-(D-1)/2}\frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)}L^{2s-D} \times \sum_{n=-\infty}^{\infty}\bigl(\nu^2 + \kappa_n^2\bigr)^{\frac{D-1}{2}-s}. \tag{2.5}$$

Here

$$\nu = mL \tag{2.6}$$

is a dimensionless parameter, and $V = \beta L L_2 \cdots L_{D-1}$ is the spacetime volume.

At this stage we convert the remaining sum in (2.5) into a contour integral. We choose our contour to be a thin rectangle enclosing the real κ-axis. We integrate a function with poles at $\kappa = \kappa_n$ whose residues are given by $(\nu^2 + \kappa_n^2)^{(D-1)/2-s}$. The natural choice leads to

$$\sum_{n=-\infty}^{\infty}\bigl(\nu^2 + \kappa_n^2\bigr)^{\frac{D-1}{2}-s} = \int_{\mathcal{C}}\frac{d\kappa}{2\pi i}\bigl(\nu^2 + \kappa_n^2\bigr)^{\frac{D-1}{2}-s}\frac{\psi'(\kappa)}{\psi(\kappa)}, \tag{2.7}$$

with $\mathcal{C}$ the contour described, and $\psi(\kappa)$ given by (2.2). For $\Re(s) > D/2$, the sides of the rectangle parallel to the imaginary axis make a vanishing contribution as we extend the rectangle to infinity. The integrand has branch points at $\kappa = \pm i\nu$. We choose the branch cuts along the imaginary axis extending from $\kappa = i\nu$