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Lorentzian geometry for detecting qubit entanglement

Joseph Samuel *, Kumar Shivam, Supurna Sinha

Raman Research Institute, Bangalore 560080, India

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ABSTRACT

We describe a new approach based on Lorentzian geometry to detect qubit entanglement. The treatment is based physically, on the causal structure of Minkowski spacetime, and mathematically, on a Lorentzian Singular Value Decomposition. A surprising feature is the natural emergence of "Energy conditions" used in Relativity. All states satisfy a "Dominant Energy Condition" (DEC) and separable states satisfy the Strong Energy Condition(SEC), while entangled states violate the SEC. We thus propose a test for two qubit entanglement which is an alternative to the positive partial transpose (PPT) test. This test is based on the partial Lorentz transformation (PLT) on individual qubits. Apart from testing for entanglement, our approach also enables us to construct a separable form for the density matrix in those cases where it exists. Our approach leads to a simple graphical three dimensional representation of the state space which shows the entangled states within the set of all states.

1. Introduction

Detecting entanglement is one of the outstanding problems in Quantum Information Theory. In two qubit systems, the Positive Partial Transpose (PPT) criterion [1–3] gives a simple, computable criterion for detecting entanglement. The criterion gives a necessary and sufficient condition for a state to be separable.

In this paper, we propose a new approach based on Partial Lorentz Transformation (PLT) of individual qubits. It turns out that like the PPT test, the PLT criterion is necessary and sufficient in the two qubit case. Here we give the PLT test as a recipe, that could be directly used by those who

* Corresponding author.
E-mail address: sam@rri.res.in (J. Samuel).

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want to apply the test. We also describe the theoretical framework behind the PLT test. In addition to showing why the test works, our Lorentzian approach yields an explicit separable form of the density matrix, when such a form exists. It also permits a complete elucidation of the state space using a Lorentzian version of the Singular Value Decomposition. The PLT test uses ideas borrowed from the space–time physics of Special Relativity.

The paper is organized as follows. In Section 2 we discuss Partial Lorentz Transformations (PLT). Section 3 describes the Lorentzian Singular Value Decomposition (LSVD) which provides the theoretical basis for the PLT test. Section 4 gives necessary and sufficient conditions on the singular values to define a state and expresses the state in separable form, under certain conditions on the singular values. We also show that these conditions are necessary for separability. Section 5 gives the test for the entanglement of a density matrix in a recipe form. This recipe can be used to detect entanglement in a two qubit system. In fact, readers interested mainly in seeing how our entanglement test works, can go directly to this section without going through the other sections where the structural framework behind the test has been elucidated. We then discuss a simple three dimensional representation of the two-qubit state space in Section 6. Section 7 deals with non generic states. We finally end the paper with some concluding remarks in Section 8.

We use a Lorentzian metric of signature mostly minus: $g = diag(1, -1, -1, -1)$. Spacetime Lorentz indices μ, ν range over 0, 1, 2, 3, as also do Frame indices $a, b, \ldots$ Both these indices are raised and lowered by the Minkowski metric and we use the Einstein summation convention. All our causal (timelike or lightlike) 4-vectors are pointing into the future. Throughout this paper, by "Lorentz group", we mean its proper, orthochronous subgroup, which preserves the time orientation as well as the spatial orientations.

2. Lorentz transformations

The states of a qubit can be expressed in space–time form by using $\sigma_\mu = (\mathbb{1}, \sigma_x, \sigma_y, \sigma_z)$, the identity and the Pauli matrices

$$\tau = u^\mu \sigma_\mu \tag{1}$$

u^μ is a real future pointing 4-vector and satisfies

$$u^\mu u^\nu g_{\mu\nu} > 0 \tag{2}$$

for impure states and

$$u^\mu u^\nu g_{\mu\nu} = 0 \tag{3}$$

for pure states. Impure states have time-like u and pure states have lightlike u. In both the cases $u^0 > 0$, the 4-vector u^μ is future pointing. If we were to fix the "normalization" by $\mathrm{Tr}(\rho) = 2$, $u^0 = 1$, the impure states can be represented in the Bloch ball $\vec{u}.\vec{u} < 1$ and the pure states on the Bloch sphere $\vec{u}.\vec{u} = 1$. All indices are raised and lowered with g. For σ^μ we have, $\sigma^\mu = g^{\mu\nu}\sigma_\nu$. The Lorentzian nature of the state space is already evident. Under Lorentz Transformations

$$u^\mu \mapsto u'^\mu = S^\mu{}_\nu u^\nu$$

where $S^\mu{}_\nu S^\alpha{}_\beta g_{\mu\alpha} = g_{\nu\beta}$. The Lorentz Transformation maps states to states. The group action has two orbits: the pure states constitute one orbit and the impure states another.

Partial Lorentz Transformations: Let ρ be a density matrix of a two qubit system. We assume ρ is non negative ($\rho \geq 0$), Hermitian ($\rho^\dagger = \rho$). In our treatment, we will not need to normalize ρ, but we suppose ρ does not vanish identically. One can expand the density matrix ρ as

$$\rho = \frac{1}{4} A^{\mu\nu} \sigma_\mu \otimes \sigma_\nu \tag{4}$$

where $A^{\mu\nu}$ can be calculated from

$$A_{\mu\nu} = \mathrm{Tr}(\rho \sigma_\mu \otimes \sigma_\nu). \tag{5}$$