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Out-of-time-order operators and the butterfly effect

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ABSTRACT

Out-of-time-order (OTO) operators have recently become popular diagnostics of quantum chaos in many-body systems. The usual way they are introduced is via a quantization of classical Lyapunov growth, which measures the divergence of classical trajectories in phase space due to the butterfly effect. However, it is not obvious how exactly they capture the sensitivity of a quantum system to its initial conditions beyond the classical limit. In this paper, we analyze sensitivity to initial conditions in the quantum regime by recasting OTO operators for many-body systems using various formulations of quantum mechanics. Notably, we utilize the Wigner phase space formulation to derive an $\hbar$-expansion of the OTO operator for spatial degrees of freedom, and a large spin $1/s$-expansion for spin degrees of freedom. We find in each case that the leading term is the Lyapunov growth for the classical limit of the system and argue that quantum corrections become dominant at around the scrambling time, which is also when we expect the OTO operator to saturate. We also express the OTO operator in terms of propagators and see from a different point of view how it is a quantum generalization of the divergence of classical trajectories.

1. Introduction

Quantum chaos attempts to generalize well-established classical diagnostics of chaos to quantum-mechanical systems [1–3]. Although classical diagnostics are well-understood, finding satisfying quantum diagnostics is an ongoing field of research [4–7]. One reason is that naïve quantum generalizations often prove unsatisfactory.

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For instance, one of the criteria for classical chaos is the sensitivity of dynamics to initial conditions. That is, for chaotic systems, two nearby initial states will diverge quickly under time evolution. This phenomenon is commonly known as the *butterfly effect*. We could try to naïvely generalize this notion to the quantum case in the following way. Given a quantum system, for an initial state $|\Psi\rangle$ and some perturbed state $|\Psi'\rangle$, we might try to define the system to be sensitive if their inner product diminishes quickly with time. This sensitivity would then be a criterion for quantum chaos. However, due to the unitarity of time evolution, the inner product actually stays constant:

$$\langle\Psi'|U^\dagger(t)U(t)|\Psi\rangle = \langle\Psi'|\Psi\rangle.$$

Hence it is not meaningful to define sensitivity by the evolution of the overlap between states.

However, this approach makes a mistake from the very start. The sensitivity criterion for classical chaos uses the difference of *observables*, such as position and momentum, as a metric for the distance between states, not the inner product in Hilbert space. The reason we obtained an unfruitful criterion for quantum chaos is because we were finding the quantum generalization of the wrong concept. Hence we need to more carefully define the classical criterion of sensitivity. To do this, we let $x(t)$ be the position of a particle at time t and x_0 its initial position. Then, we say the system exhibits the butterfly effect if

$$\left|\frac{\partial x(t)}{\partial x_0}\right| \sim e^{\lambda t}\,, \tag{1}$$

where $\lambda > 0$ is known as a *Lyapunov exponent*. To generalize this notion to quantum systems, we first re-write the sensitivity as a Poisson bracket:

$$\frac{\partial x(t)}{\partial x_0} = \{x(t),\ p_0\}_{\text{Poisson}}\,, \tag{2}$$

where p_0 is the initial momentum. We can now proceed by canonical quantization to obtain the quantity

$$\frac{1}{i\hbar}[\widehat{x}(t), \widehat{p}].$$

We are interested in the magnitude of this operator, so we take its norm squared:

$$\frac{1}{i\hbar}[\widehat{x}(t), \widehat{p}] \cdot -\frac{1}{i\hbar}[\widehat{x}(t), \widehat{p}]^\dagger = -\frac{1}{\hbar^2}[\widehat{x}(t), \widehat{p}]^2. \tag{3}$$

The above is an example of an out-of-time-order (OTO) operator, which first appeared in [8] and is related to the Loschmidt echo studied in [9–13]. It is so named since it contains terms that are not time-ordered such as $\widehat{x}(t)\widehat{p}\,\widehat{x}(t)\widehat{p}$. Note that in recent literature, any operator of the form $[\widehat{W}(t), \widehat{V}] \cdot [\widehat{W}(t), \widehat{V}]^\dagger$ is referred to as an OTO operator (see, for instance, [14,15]). The corresponding many-body, higher dimensional version of Eq. (3) is given by

$$-\frac{1}{\hbar^2}\left[\widehat{x}_i(t), \widehat{p}_j\right]^2, \tag{4}$$

where x_i, p_j are the position and momentum of the ith and jth coordinate, respectively. Eq. (4) quantifies the sensitivity of the position of the ith coordinate to the initial position of the jth coordinate.

We can then define a quantum system to be sensitive to initial conditions if the OTO operator grows exponentially with time. However, we obtained the OTO from canonically quantizing a quantity measuring sensitivity of classical trajectories. Intuitively, we might then expect that the OTO is measuring the sensitivity of some form of quantum trajectory, but it is not clear what this means. By the correspondence principle, the OTO must reduce to the sensitivity in the classical limit, but it is not obvious what the OTO is measuring beyond this limit. Indeed, the OTO is known to saturate, that is, deviate from exponential growth, by a time scale known as the scrambling or Ehrenfest time, while $\left|\frac{\partial x(t)}{\partial x_0}\right|$ of a chaotic classical system should grow exponentially indefinitely. Then, in what precise sense is the OTO is measuring a quantum analogue of the butterfly effect?