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Landau level quantization of Dirac electrons on the sphere

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Interactions in Landau levels can stabilize new phases of matter, such as fractionally quantized Hall states. Numerical studies of these systems mostly require compact manifolds like the sphere or a torus. For massive dispersions, a formalism for the lowest Landau level on the sphere was introduced by Haldane (1983). Graphene and surfaces of 3D topological insulators, however, display massless (Dirac) dispersions, and hence require a different description. We generalize a formalism previously developed for Dirac electrons on the sphere in zero field to include the effect of an external, uniform magnetic field.

Progress in theoretical physics has always been achieved through the interplay of obtaining experimental data with comparing it to the predictions of the ideas, concepts, and theories suggested to explain the data. In earlier periods, the implications of theoretical models could be explored only through analytic calculations. During the past four decades, however, the availability of ever more powerful computers has significantly reshaped this process. Among early highlights were the development of the renormalization group by Wilson [1], the discovery of universality in the onset of chaos by Feigenbaum [2], and the formulation of Laughlin's wave function for fractional quantized Hall liquids [3]. Laughlin's discovery is particularly striking in this context as it was guided by a numerical experiment [4]. Laughlin numerically diagonalized a system of a few electrons in the lowest Landau level in the open plane, and observed that the canonical angular momentum of the ground state jumped by a factor of three upon turning on a strong repulsive interaction. The experimental discovery of the effect had inspired the numerical experiment, and the numerical experiment provided the crucial hint to the formulation of the theory. The theory was only accepted by the community at large after Haldane formulated it on a sphere [5], a geometry without a boundary and hence without gapless edge modes, and showed that Laughlin's trial state can be adiabatically connected to the

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ground state for Coulomb interactions without closure of the energy gap [6]. A more recent example for the importance of numerical experiments is the discovery of the topological insulator (TI) as a consequence of band inversion by Kane and Mele [7,8], a phase which was subsequently realized in HgTe quantum wells [9,10].

The efficient implementation of numerical experiments often requires geometries which cannot be realized in a laboratory, such as periodic boundary conditions (PBCs). When the underlying lattice plays no role in the effective model one wishes to study, the simplest geometry without a boundary is the sphere. It continues to be of seminal importance in numerical studies of quantized Hall states and other states of matter in two dimensional electron gases subject to a magnetic field. While most of the work on TIs focusses on the single particle description of topologically non-trivial band structures, the most promising avenues to observe topologically non-trivial many body condensates in this context may be at the surface of a 3D TI [11,12]. The single particle states on these surfaces are described by a single Dirac cone, which would be impossible to realize on a lattice due to the fermion doubling theorem [13]. Even as a continuum theory, coupling the electrons minimally to the electromagnetic gauge field requires an even number of Dirac cones, or an axion term on one side of the surface [14]. In other words, a single Dirac cone at a surface requires a termination of a topological insulator [15]. The situation is less intricate in graphene, where a 2D lattice not embedded in a 3D topological structure features one Dirac cones per spin and valley degree of freedom, and hence a total of four cones [16].

Regarding the numerical study of interaction effects on surfaces of 3D TIs, the only work published so far has employed a spherical geometry [17,18]. (For PBCs, the numerics is far more challenging, and the studies performed so far are unpublished as of yet [19].) To formulate the single particle Hilbert space for the single Dirac cone on the sphere, we employed a formalism introduced earlier by one of us [20] to describe Landau levels (LLs) for massive electrons on the sphere, which in turn generalized the spinor coordinate formalism introduced earlier by Haldane [5] for the lowest LL. The magnetic monopole in the center of the sphere, of monopole charge $2s_0 = +1$ for $\uparrow$ spins and $2s_0 = -1$ for $\downarrow$ spins, emerges from the Berry's phase associated with rotations of the reference system for the spin. (In our notation, spin $\uparrow$ and $\downarrow$ refer to spin directions normal to the surface of the sphere.) We obtained the single particle Hamiltonian,

$$H = \frac{\hbar v}{R}\begin{pmatrix} 0 & -S^+ \\ -S^- & 0 \end{pmatrix}, \tag{1}$$

where the angular momentum operators S^- and S^+ effectively act as LL "raising" and "lowering" operators on the sphere, v is the Dirac velocity, and R the radius of the sphere. This form resembles the single particle Hamiltonian for Dirac electrons subject to a uniform magnetic field $\boldsymbol{B} = -B\boldsymbol{e}_z$ in the plane,

$$H = \frac{\hbar v\sqrt{2}}{l}\begin{pmatrix} 0 & \mathrm{i}a^\dagger \\ -\mathrm{i}a & 0 \end{pmatrix}, \tag{2}$$

where $a^\dagger$ and a are Landau level raising and lowering operators (see Refs. [21] or [22] for reviews of the formalism), and $l = \sqrt{\frac{\hbar c}{eB}}$ is the magnetic length.

In this paper, we first provide a more detailed derivation of (1) than space allowed in Ref. [17], and second, show that (1) also holds in the presence of an external, radial magnetic field $\boldsymbol{B} = B\boldsymbol{e}_r$ supplementing the Berry flux. We assume a field strength $B = 2b_0\Phi_0/4\pi R^2$, where $\Phi_0 = 2\pi\hbar c/e$ with $e > 0$ is the Dirac flux quantum, such that the total number of Dirac flux quanta through the surface is $2b_0$. *The only change due to the field is that the* $\uparrow$ *and* $\downarrow$ *spin components of the spinor* ψ^λ_{nm},

$$H\psi^\lambda_{nm} = E_n\psi^\lambda_{nm}, \quad \psi^\lambda_{nm} = \begin{pmatrix} \phi^\uparrow_{nm} \\ \lambda\phi^\downarrow_{nm} \end{pmatrix}, \tag{3}$$

are given by (massive) LL wave functions [20] corresponding to total magnetic flux $\Phi = (2b_0 \pm 1)\Phi_0$ *rather than just* $\pm\Phi_0$ *through the surface of the sphere*, and the energies for states in (Dirac) LL n are given by

$$E_n = \lambda\frac{\hbar v}{R}\sqrt{(2b_0 + n)n} \tag{4}$$