Annals of Physics 394 (2018) 40–49

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Two patterns of $\mathcal{PT}$-symmetry breakdown in a non-numerical six-state simulation

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HIGHLIGHTS

- Hot topic (PT symmetry breakdown in quantum theory) studied.
- Next to elementary toy model considered.
- Characteristic features (like quantum phase transitions) sampled.
- Surprisingly, the results are non-numerical.
- The method (inverse-function treatment) is brand new.

ARTICLE INFO

Article history:
Received 7 December 2017
Accepted 19 April 2018
Available online 27 April 2018

Keywords:
Quantum theory
Non-Hermitian observables
Bound state instabilites
Typology
Discrete models

ABSTRACT

Three-parametric family of non-Hermitian but $\mathcal{PT}$-symmetric six-by-six matrix Hamiltonians $H^{(6)}(x, y, z)$ is considered. The $\mathcal{PT}$-symmetry remains spontaneously unbroken (i.e., the spectrum of the bound-state energies remains real so that the unitary-evolution stability of the quantum system in question is shown guaranteed) in a non-empty domain $\mathcal{D}^{(physical)}$ of parameters x, y, z. The construction of the exceptional-point (EP) boundary $\partial\mathcal{D}^{(physical)}$ of the physical domain is preformed using an innovative non-numerical implicit-function-construction strategy. The topology of the resulting EP boundary of the spontaneous $\mathcal{PT}$-symmetry breakdown (i.e., of the physical "horizon of stability") is shown similar to its much more elementary $N = 4$ predecessor. Again, it is shown to consist of two components, viz., of the region of the quantum phase transitions of the first kind (during which at least some of the energies become complex) and of the quantum phase transitions of

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https://doi.org/10.1016/j.aop.2018.04.023

the second kind (during which some of the level pairs only cross but remain real).

1. Introduction

In 1998, Bender with Boettcher [1] conjectured that the reality of the bound state energy spectra (i.e., the unitarity of the evolution) might be attributed to the unbroken $\mathcal{PT}$-symmetry (i.e., parity times time-reversal symmetry) of the underlying phenomenological Hamiltonian H. Mathematical formulation as well as implementations of the newly developed theory were, twelve years later, reviewed and summarized by Mostafazadeh [2]. At present it is widely accepted that the manifest non-Hermiticity of the $\mathcal{PT}$-symmetric Hamiltonians with real spectra is fully compatible with the Stone's theorem [3] and with the unitarity of the evolution of the quantum system in question [4].

The price to pay for the resolution of the apparent paradox lies in the necessity of an *ad hoc* amendment of the Hilbert space. Simply stated (see, e.g., [5]), one has to distinguish between the naively preselected initial, unphysical, "friendly but false" Hilbert space $\mathcal{H}^{(F)}$ (in which our $\mathcal{PT}$-symmetric Hamiltonian with real bound-state spectrum appears manifestly non-Hermitian, $H \neq H^\dagger$) and its "standard physical" amendment $\mathcal{H}^{(S)}$ (here, the inner product is amended in such a way that the *same* operator becomes self-adjoint, $H = H^\ddagger$).

The innovative picture of quantum dynamics led to a perceivable extension of the class of tractable quantum Hamiltonians. For example, in the traditional unitary quantum theory of textbooks the linear differential Hamiltonians

$$H = -\triangle + V(\vec{x}) \tag{1}$$

must be kept self-adjoint in $\mathcal{H}^{(S)} = \mathcal{H}^{(F)} = L^2(\mathbb{R}^d)$. In the new context the constraint was relaxed. The progress was rendered possible by the separation of $\mathcal{H}^{(F)} = L^2(\mathbb{R}^d) \neq \mathcal{H}^{(S)}$. This resulted in the representation of unitary systems in two different Hilbert spaces, viz., in physical $\mathcal{H}^{(S)}$ and, simultaneously, in auxiliary unphysical $\mathcal{H}^{(F)}$. A number of innovative model-building activities followed [6].

Successfully, the mathematical meaning of $\mathcal{PT}$-symmetry $H\mathcal{PT} = \mathcal{PT}H$ was identified with the older concepts of pseudo-Hermiticity $H^\dagger\mathcal{P} = \mathcal{P}H$ [2] *alias* Krein-space self-adjointness [7] of the Hamiltonian. Still, for the generic non-Hermitian Hamiltonians the physical essence of quantum dynamics in $\mathcal{H}^{(S)}$ appeared counterintuitive and deeply non-local [8]. It has been revealed that for many non-Hermitian local potentials $V(\vec{x})$ (some of which played the role of benchmark toy models) the amended physical Hilbert space $\mathcal{H}^{(S)}$ need not exist, in mathematical sense, at all [9].

One of the ways out of the crisis has been found in a return to the more restricted class of the so called quasi-Hermitian Hamiltonians H. In nuclear physics, for example, these operators were obligatorily assumed bounded in $\mathcal{H}^{(F)}$ [10]. Often, they were even represented by the mere finite, N-dimensional matrices $H^{(N)}$. In what follows, we shall also proceed along this line.

In the historical perspective [11] the inspiration of the latter strategy can be traced back to the Kato's rigorous mathematical monograph [12]. Many illustrative Hamiltonians were chosen there in the form of matrices with minimal dimension $N = 2$. Also in Ref. [13] devoted to the study of several manifestly non-Hermitian differential operators (1), several anomalous spectral features caused by $\mathcal{PT}$-symmetry were successfully mimicked by certain most elementary benchmark matrices $H^{(N)}$ with $N = 2$.

Due to $\mathcal{PT}$-symmetry, the bound-state-energy spectrum of $H^{(N)}$ can be either "physical" (i.e., real, compatible with unitarity) or "unphysical" (i.e., containing one or several non-real, complex conjugate pairs). This leads to one of the most interesting mathematical questions and challenges in the $\mathcal{PT}$-symmetric quantum mechanics: Once we assume that a given $\mathcal{PT}$-symmetric Hamiltonian depends on a $J-$plet of couplings or dynamics-specifying real parameters $g_1 = a, g_2 = b, \ldots, g_J = z$, we must be able to separate the Euclidean space $\mathbb{R}^J$ of these parameters into an open domain $\mathcal{D}^{(N)}_{(physical)}$ (in