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A general asymptotic formula for distinct partitions

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ABSTRACT

Many asymptotic formulas exist for unrestricted integer partitions as well as for equal partitions of integers into a finite number of parts. We use an analogy with fermion gases and the tools of statistical physics to derive asymptotic formulas for distinct partitions with a large but finite number of parts. These results are supported by the fact that we recover some other existing asymptotic results and by numerical comparisons with exact results.

0. Introduction

The integer partition problem has generated a whole set of literature and mathematical techniques. Theoretical physicists have contributed to this literature because of the analogy of quantum states counting with partition counting: there is a mapping between the statistical properties of boson and fermion gases on the one hand and the counting of equal and distinct partitions on the other hand.

In this paper, we focus on the counting of distinct partitions and we use the analogy with fermion gases to get the asymptotic behavior of the number of distinct partitions.

A partition of a positive integer E is a way of writing E as a sum of positive integers (called parts or summands), where the order of the summands does not matter. For instance, the number of partitions of the integer 5 is equal to 7 because there are 7 ways to write 5 as a sum of positive integers:

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 \tag{1}$$

These partitions are called equal partitions because the summands may be equal to each other. Partitions into distinct parts are called distinct partitions. Other constraints may be added, for instance

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when the number of parts is given or when the summands are constrained to be smaller than a given value. These partitions are called restricted partitions. In the case of the integer 5, there are only three distinct partitions:

$$5 = 4 + 1 = 3 + 2 \tag{2}$$

and there is only one such distinct partition with parts not larger than 3 which is $5 = 3 + 2$.

An asymptotic formula for the number of simple partitions as the one of Eq. (1) was given by [1]:

$$p(E) \sim \frac{1}{4\sqrt{3}E} e^{\pi\sqrt{2E/3}} \tag{3}$$

In the case of distinct partitions, there is a similar asymptotic formula:

$$q(E) \sim \frac{1}{4.3^{1/4}.E^{3/4}} e^{\pi\sqrt{E/3}} \tag{4}$$

Many papers in the literature provide asymptotic expressions for restricted partitions. In a famous paper, Erdős and Lehner [2], and then Szekeres [3,4], compute the asymptotic expression for the number $P(E, N)$ of integer partitions of an integer E in maximum N parts. We are not aware of a similar asymptotic formula for the number of restricted distinct partitions $q(E, N)$. We know that this number is linked to the number of restricted equal partitions $P(E, N)$ as illustrated for instance in Comtet et al. [5,6]; in particular, they show that the asymptotic behavior of both functions are significantly different for large E.

When the size of the parts are themselves restricted to be lower than a given integer B, the amount of available literature is more limited. In [7], Almkvist and Andrews provide a Hardy–Ramanujan series for restricted equal partitions. Ratsaby [8] provide an asymptotic expression for the number of compositions of E in N positive integer parts under the constraint that each part is not larger than B. Compositions are equal partitions for which we count all the ordered partitions as different partitions, even if the parts are equal (for instance $3 = 1 + 2 = 2 + 1$ count for 2 compositions).

This paper focuses on non ordered restricted distinct partitions. Let $q(E, N)$ denote the number of partitions of an integer E into N distinct integer parts. By using tools of statistical physics, we obtain an explicit asymptotic formula for the function $q(E, N)$ when both E and N are large, with $N^2/E \to u$. Additionally, we use the same approach to compute an approximation of the asymptotic number $q(E, N, B)$ of distinct partitions with N terms each one being smaller than a given value B, in the limit when these three variables go to infinity but with constant ratios $N^2/E = u$ and $N/B = p$. These constrained ratios correspond to concrete situations in machine learning and in some applications to credit scoring [9].

The plan of the paper is as follows: we first detail the analogy with statistical physics systems and distinct partitions. Second, we derive the asymptotic formula for the number of distinct partitions $q(E, N)$ and show by numerical computations that this asymptotic approximation is very accurate. Finally, we extend the formula to cases where, in addition, the summands constrained to be smaller or equal to a given integer B; we give some insight on the accuracy of the saddle-point approximation of the function $q(E, N, B)$.

1. Methodology and notations

The analogy between physical systems and integer partitions is an old problem and is described in [10] and [11] for instance. For a short review, we refer to [12] as well. Throughout this paper, we follow a methodology similar to the one of [11]. We implement it in Section 2 with the function $q(E, N)$ and in Section 3 with the function $q(E, N, B)$. We first introduce the partition function related to the function $q(E, N)$:

$$Z_D(x, z) = \sum_{E=1}^{\infty}\sum_{N=1}^{\infty} z^N x^E q(E, N) \tag{5}$$