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Boundaries without boundaries

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ABSTRACT

Starting with a quantum particle on a closed manifold without boundary, we consider the process of generating boundaries by modding out by a group action with fixed points, and we study the emergent quantum dynamics on the quotient manifold.

As an illustrative example, we consider a free nonrelativistic quantum particle on the circle and generate the interval via parity reduction. A free particle with Neumann and Dirichlet boundary conditions on the interval is obtained, and, by changing the metric near the boundary, Robin boundary conditions can also be accommodated. We also indicate a possible method of generating non-local boundary conditions.

Then, we explore an alternative generation mechanism which makes use of a folding procedure and is applicable to a generic Hamiltonian through the emergence of an ancillary spin degree of freedom.

1. Introduction

Boundary conditions emerge as a model of the interaction of a confined physical system with its boundary. In this paper we will be interested in applications to quantum systems. Quantum boundary conditions can be effectively used to describe different quantum situations ranging from topology change in quantum gravity [1,2], to the Casimir effect [3] in quantum field theory and the quantum Hall effect in condensed matter physics [4]. For general reviews see [5–7].

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From the mathematical point of view quantum boundary conditions emerge as a parametrization of the self-adjoint extensions of a differential operator [8]. It is well known, indeed, from the basic principles of quantum mechanics, that physical observables correspond to self-adjoint operators [9,10].

A paradigmatic example is a free nonrelativistic quantum particle in a cavity Ω, an open bounded set of $\mathbb{R}^n$, whose kinetic energy is described by the Laplace operator:

$$H = -\frac{\hbar^2}{2m}\Delta, \qquad D(H) = C_c^\infty(\Omega), \tag{1}$$

where m is the mass of the particle, $\hbar$ is the Planck constant, and the operator domain $D(H)$ is the space of smooth functions compactly supported in Ω. This operator is symmetric but not self-adjoint, and admits infinitely many self-adjoint extensions as provided by von Neumann's theory of self-adjoint extensions [11].

There has been an increasing interest in classifying the self-adjoint extensions of elliptic operators in terms of boundary conditions. In particular, it was proved [5,12,13] that the set of the self-adjoint extensions of the Laplace operator on a manifold with boundary is in one-to-one correspondence with the set of unitary operators on the boundary. The situation can be easily specialized for the one dimensional case.

Consider the interval $\Omega = (0, \pi)$ and let $L^2(\Omega)$ be the Hilbert space of square integrable functions on Ω. The Hamiltonian (1) reads

$$H = \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}. \tag{2}$$

As already stressed, this operator is not self-adjoint, but admits infinitely many self-adjoint extensions, each of which is parametrized by a two dimensional unitary matrix [5]. Well-known boundary conditions are Dirichlet:

$$\psi(0) = 0, \qquad \psi(\pi) = 0, \tag{3}$$

and Neumann:

$$\psi'(0) = 0, \qquad \psi'(\pi) = 0, \tag{4}$$

where $\psi' = \mathrm{d}\psi/\mathrm{d}x$. These are examples of local boundary conditions, which do not mix the values at the endpoints of the interval. The most general local boundary conditions are given by Robin:

$$\psi'(0) = \mu_0\psi(0) \qquad \psi'(\pi) = -\mu_\pi\psi(\pi), \qquad \mu_0, \mu_\pi \in \mathbb{R}. \tag{5}$$

Notice that for $\mu = 0$ one recovers Neumann, while for $\mu \to \infty$ one gets Dirichlet. In general, Robin boundary conditions mix the values of the function ψ with that of its derivative ψ' at the boundary points $x = 0$ and $x = \pi$.

A family of non-local boundary conditions is provided by

$$\psi(0) = e^{i\alpha}\psi(\pi), \qquad \psi'(0) = e^{i\alpha}\psi'(\pi), \qquad \alpha \in \mathbb{R}, \tag{6}$$

which are known as twisted (or pseudo-) periodic boundary conditions. As a particular case, one recovers periodic and anti-periodic boundary conditions for $\alpha = 0$ and $\alpha = \pi$, respectively.

Several problems can be studied with the above technology ranging from one dimensional systems with time-dependent boundary conditions [14–16] to quantized fields [17–22].

The central question of interest in this paper is the following: *Can one generate boundary conditions starting from a quantum system on a manifold without boundaries?*

In the first part of the paper we will analyze what kind of boundary conditions can emerge via a symmetry reduction procedure starting from a manifold without boundary. Then, in the second part, we will explore an alternative path for the same problem that makes use of a folding procedure.

The paper is organized as follows. In Section 2 we consider the specific case of a nonrelativistic particle on the circle $\mathbb{S}$, and explicitly show how one can generate boundary conditions by the action