Annals of Physics 393 (2018) 327–334

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Fractional Ornstein–Uhlenbeck noise

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HIGHLIGHTS

- Calculation of the correlation function for the fractional Ornstein–Uhlenbeck noise.
- Calculation of the mean energy of the harmonic oscillator driven by the fractional Ornstein–Uhlenbeck noise.
- The model can be applied to the RLC circuit driven by a supercapacitor with the white noise.

ARTICLE INFO

Article history:
Received 30 October 2017
Accepted 14 April 2018
Available online 21 April 2018

Keywords:
Langevin equation
Fractional Ornstein–Uhlenbeck noise
Supercapacitor
Harmonic oscillator
Colored noise

ABSTRACT

Fractional Ornstein–Uhlenbeck noise is considered and investigated. The fractional Ornstein–Uhlenbeck noise may be linked with a supercapacitor driven by the white noise, and its correlation function for the stationary state shows monotonic and oscillatory decays. In the case of the oscillatory behavior the correlation function presents behaviors similar to those of the harmonic noise (harmonic oscillator driven by the white noise). For application, the Langevin equation with the harmonic potential driven by the fractional Ornstein–Uhlenbeck noise is considered; the first two moments and mean energy are investigated.

1. Introduction

Fractional calculus is a ubiquitous tool due to the fact that it has been applied to a wide range of systems, such as physical, chemical and biological systems. For instance, it has been used to model diffusion phenomena [1], tissue viscoelasticity [2,3], electronic circuits [4,5], biological systems [6–13] and complex phenomena [14,15].

The well-known example of a diffusion process is the Brownian motion. Besides, diffusion processes are classified according to their mean square displacements; for normal diffusion the mean square displacement grows linearly with time, and in other situations the processes are said to

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https://doi.org/10.1016/j.aop.2018.04.019
0003-4916/

exhibit anomalous diffusion. Nowadays, there are several approaches to describe anomalous diffusion processes, and they have been employed to describe many different systems [1,16–22]. One of the most striking features incorporated into these approaches is the memory effect. In particular, the memory effect incorporated into the Langevin approach, referred to as generalized Langevin equation (GLE) [22–26], which includes fractional approach, can be associated to the retardation of friction and fractal media [27–29]. Besides, anomalous diffusion processes have been observed in a lot of systems, for instance, bacterial cytoplasm motion [30], conformational fluctuations within a single protein molecule [31] and fluorescence intermittency in single enzymes [32].

The aim of this work is to investigate a fractional Ornstein–Uhlenbeck (FOU) noise in which the ordinary derivative is replaced by a fractional derivative. In this approach the procedure to deal with a colored noise is to extend the space of variables so that the noise itself becomes a variable driven by the white noise. However, it should be noted that two generalizations of the ordinary Ornstein–Uhlenbeck process have appeared in the literature, and they have been named as the fractional Ornstein–Uhlenbeck process. One of them generalizes the ordinary Ornstein–Uhlenbeck process by replacing the ordinary derivative by a fractional derivative (see for instance Refs. [27–29,33,34] for description and application). The other one generalizes the ordinary Ornstein–Uhlenbeck process driven by a fractional Brownian motion (see for instance Refs. [35–38] for description and application). In order to connect the fractional Ornstein–Uhlenbeck noise with the fractional Ornstein–Uhlenbeck process the stationary state of the correlation function should be demonstrated. Besides, the FOU process has been applied to financial time series [28,29] and it may also be linked with a supercapacitor driven by the white noise. In the next section the FOU noise is introduced and its correlation function for the stationary state is shown numerically. In Section 3 the harmonic oscillator driven by the fractional Ornstein–Uhlenbeck noise is considered. The first two moments and mean energy are analyzed. Conclusion is provided in Section 4.

2. Fractional Ornstein–Uhlenbeck noise

The fractional Ornstein–Uhlenbeck process is a generalization of the ordinary Ornstein–Uhlenbeck process in which the ordinary derivative is replaced by a fractional derivative, and it is described by the following Langevin equation:

$$\frac{d^\alpha x}{dt^\alpha} + \Omega_\alpha x = \beta_\alpha L(t), \tag{1}$$

where Ω_α and β_α are real positive numbers, and $d^\alpha x/dt^\alpha$ is the Caputo fractional derivative defined by

$$\frac{d^\alpha x}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t d\tau \frac{\frac{d^n x}{d\tau^n}}{(t-\tau)^{\alpha+1-n}}, \quad n-1 < \alpha < n, \tag{2}$$

$\Gamma[z]$ is the gamma function and $L(t)$ is the Gaussian white noise with the ensemble averages given by

$$\langle L(t) \rangle = 0 \tag{3}$$

and

$$\langle L(t_1) L(t_2) \rangle = q\delta(t_1 - t_2), \tag{4}$$

where q is a real positive value and it is related to the intensity of the noise.

Hereafter the value of α is restricted to the interval $0 < \alpha < 2$. In fact, there are various definitions of fractional derivative. One has chosen the Caputo fractional derivative in Eq. (1) due to the fact that the FOU process described by it can be associated with the output signal of a supercapacitor [4].

Eq. (1) can be solved by employing the Laplace transform, and the result is given by

$$x(t) = \langle x \rangle + \beta_\alpha \int_0^t d\tau (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\Omega_\alpha (t-\tau)^\alpha\right) L(\tau), \tag{5}$$