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Algebraic calculations for spectrum of superintegrable system from exceptional orthogonal polynomials

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ABSTRACT

We introduce an extended Kepler–Coulomb quantum model in spherical coordinates. The Schrödinger equation of this Hamiltonian is solved in these coordinates and it is shown that the wave functions of the system can be expressed in terms of Laguerre, Legendre and exceptional Jacobi polynomials (of hypergeometric type). We construct ladder and shift operators based on the corresponding wave functions and obtain their recurrence formulas. These recurrence relations are used to construct higher-order, algebraically independent integrals of motion to prove superintegrability of the Hamiltonian. The integrals form a higher rank polynomial algebra. By constructing the structure functions of the associated deformed oscillator algebras we derive the degeneracy of energy spectrum of the superintegrable system.

1. Introduction

Many families of exceptional orthogonal polynomials have been successfully used to construct new superintegrable systems, higher order integrals of motion and higher order polynomial algebras [1–5]. In this paper, we use the recurrence approach to extend the three parameters Kepler–Coulomb system [6].

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The exceptional orthogonal polynomials (EOP) were first explored in [7,8]. These polynomials form complete, orthogonal systems extending the classical orthogonal polynomials of Hermite, Laguerre and Jacobi. More recently much research has been done extending the theory of EOPs in various directions in mathematics and physics, in particular, exactly solvable quantum mechanical problems for describing bound states [9–17] and scattering states [18–21], diffusion equations and random processes [22–24], quantum information entropy [25], exact solutions to Dirac equation [26], Darboux transformations [14,15,27–31] and finite-gap potentials [32]. Recent progress has been made constructing systems relating superintegrability and supersymmetric quantum mechanics with exceptional orthogonal polynomials [1,33].

The research for superintegrable systems with second-order integrals in conformally flat spaces started in the mid sixties [34]. Over the last decade the topic of superintegrability has become an attractive area of research as these systems possess many desirable properties and can be found throughout various subjects in mathematical physics. For a detailed list of references on superintegrability, we refer the reader to the review paper [35]. One systematic approach to superintegrability is to derive spectra of 2D superintegrable systems based on quadratic and cubic algebras involving three generators [36–38]. In particular, the method of realization in the deformed oscillator algebras [39] has been effective for obtaining finite dimensional unitary representations [37,40]. In fact, this approach was extended to classes of higher order polynomial algebras with three generators [41] as well as higher rank polynomial algebras of superintegrable systems in higher dimensional spaces [42,43]. However, it is quite involved to apply the direct approach to obtain the corresponding polynomial algebras, Casimir operators and deformed oscillator algebras.

These difficulties can be overcome using a constructive approach based on eigenfunctions of the models. This approach is a useful tool to construct well-defined integrals of motion in classical and quantum mechanical problems. Many papers were devoted to construct integrals of motion and their corresponding higher order symmetry algebras based on lower-(first and second) ones [34,44–47] and higher-order ladder operators [4,5,48–54] in various aspects. In fact, the constructive approach has shown a close connection with special functions and (exceptional) orthogonal polynomials [1,33,55–59].

The search of exactly solvable generalizations of hydrogen like models took various directions [60–65]. Such models can be used in context of ring shaped molecules [60] and in the study of quantum scattering [62]. A classification of superintegrable systems preserving rotational invariance and involving spin interaction was pursued [63–65]. In this paper, we introduce a new exactly solvable Hamiltonian system in 3D, which is a singular deformation of the Coulomb potential. Its wave functions are given as products of Laguerre, Legendre and exceptional Jacobi polynomials. We show that the system is superintegrable by constructing higher order integrals of the motion using the recurrence relation approach. The symmetry algebra generated by these conserved quantities enables us to give an algebraic derivation for the energy spectrum.

The paper is organized as follows: in Section 2, we present a new Hamiltonian system in 3D and show that its Schrödinger wave functions can be expressed in terms of Laguerre, Legendre and exceptional polynomials and obtain its physical spectra. In Section 3, we construct a set of ladder and shift operators based on the wave functions and show that their suitable combination give the integrals of motion, thus proving the superintegrability of the model. We present the higher rank polynomial algebra generated by these integrals and the realization of this symmetry algebra in terms of the deformed oscillator algebra. By constructing finite-dimensional unitary representation of the symmetry algebra, we obtain the energy spectrum of superintegrable system.

2. Extended Kepler–Coulomb system

Consider the generalization of the three parameter Kepler–Coulomb Hamiltonian [6] in the spherical coordinates

$$H = \frac{1}{2}\mathbf{p}^2 - \frac{\alpha}{2r} + \frac{1}{2r^2\sin^2\theta}\left[\frac{\gamma^2-\frac{1}{4}}{4\sin^2\frac{\phi}{2}} + \frac{\delta^2-\frac{1}{4}}{4\cos^2\frac{\phi}{2}} + \frac{2(1-b\cos\phi)}{(b-\cos\phi)^2}\right], \tag{2.1}$$