Annals of Physics 388 (2018) 12–25

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Exact sum rules for quantum billiards of arbitrary shape

Paolo Amore

Facultad de Ciencias, CUICBAS, Universidad de Colima, Bernal Díaz del Castillo 340, Colima, Colima, Mexico

HIGHLIGHTS

- Sum rules of order one for domains of arbitrary shape are considered.
- The sum rules are proved to be finite.
- Exact expressions for number of examples are obtained.
- These expressions are verified with accurate numerical tests.

ARTICLE INFO

Article history:
Received 18 July 2017
Accepted 2 November 2017
Available online 7 November 2017

Keywords:
Spectral sum rule
Quantum billiard
Weyl's law

ABSTRACT

We have derived explicit expressions for the sum rules of order one of the eigenvalues of the negative Laplacian on two dimensional domains of arbitrary shape. Taking into account the leading asymptotic behavior of these eigenvalues, as given from Weyl's law, we show that it is possible to define sum rules that are finite, using different prescriptions. We provide the explicit expressions and test them on a number of non trivial examples, comparing the exact results with precise numerical results.

1. Introduction

We consider the Helmholtz equation on a two dimensional region Ω

$$-\Delta\Psi_n = E_n\Psi_n \quad ; \quad n = 1, 2, \ldots \tag{1}$$

where E_n are the eigenvalues and Ψ_n the eigenfunctions, obeying appropriate boundary conditions on $\partial\Omega$.[1]

E-mail address: paolo@ucol.mx.

[1] In our discussion the boundary conditions will either be Dirichlet ($\Psi_n|_{\partial\Omega} = 0$) or Neumann ($\hat{n}\cdot\nabla\Psi_n\big|_{\partial\Omega} = 0$, where $\hat{n}$ is the unit vector normal to the border at each point).

https://doi.org/10.1016/j.aop.2017.11.003

Eq. (1) can be solved exactly only in special cases such as for the circle or the rectangle and therefore one must rely on approximations for more general cases. For instance, if the domain Ω is a perturbation of a circle (or of any other domain where the exact solutions are known), perturbation theory (PT) allows one to obtain explicit expressions for the eigenvalues and eigenfunctions of Eq. (1) as a power series in the perturbation parameter; when the domain Ω is not a perturbation, the equation can still be solved numerically, for a limited portion of the spectrum, using the different techniques. Variational estimates can also be used to provide rigorous bounds on specific eigenvalues (in particular on the lowest eigenvalue).

The asymptotic behavior of the spectrum, on the other hand, is described by Weyl's law, which relates the counting function $N(E) = \# \{E_n \leq E\}$ to the geometric properties of the domain

$$N(E) = \frac{A}{4\pi} E \mp \frac{L}{4\pi}\sqrt{E} + o(\sqrt{E}) \quad ; \quad E \to \infty \tag{2}$$

where A and L are respectively the area and perimeter of the domain. The $\mp$ sign refers to Dirichlet/Neumann boundary conditions.

A useful strategy in dealing with Eq. (1) is to apply a conformal transformation, mapping the original region into a suitable region, where a complete set of eigenfunctions of the Laplacian is known (the existence of this map is granted by Riemann's mapping theorem, although finding the explicit expression may be difficult).

In this case the Helmholtz equation is transformed to the equation (see for example Ref. [1])

$$-\Delta\Psi_n = E_n \Sigma(x, y)\Psi_n \tag{3}$$

where Σ is a "density" related to the conformal transformation which maps Ω to the circle (or to any suitable region). The solutions to this equation can be approximated using either analytical or numerical methods: for example, when the map is a perturbation of the identity one can apply perturbation theory (see Ref. [2]), whereas in more general cases, spectral (Rayleigh–Ritz) [3] or pseudospectral (collocation) [2] methods can be used to obtain precise results.

Itzykson and collaborators [4] have first noticed that it is possible to obtain explicit (exact) expressions for the sum rules of the eigenvalues of Eq. (3)

$$\zeta(s) = \sum_{p=1}^{\infty} \frac{1}{E_p^s} \quad ; \quad s = 2, 3, \ldots \tag{4}$$

calculating the traces of the appropriate operators, without the need of explicitly knowing the eigenvalues E_n. Eq. (6) of Ref. [4] deals with the special case $s \to 1^+$, for which $\zeta(s)$ diverges, identifying the leading contributions. Kvitsinsky [5] has later applied similar ideas to discuss the spectral sum rules of nearly circular domains, discussing in particular the case of regular n-sided polygonal domains (note that the analysis of Refs. [4] and [5] is limited to Dirichlet boundary conditions).

Dittmar [6] has obtained explicit formulas for the sum rules of order two, both for Dirichlet and Neumann boundary conditions, for simply connected domains of the plane, using a conformal transformation of the original domain to the unit disk. Examples of sums rules of order two for the cardioid and related domains are given in Ref. [7]. Dostanic has obtained an expression for the regularized trace of the Dirichlet laplacian Ref. [8].

More recently we have derived the general expressions for the spectral sum rules of inhomogeneous strings and membranes (in one and two dimensions), of which Eq. (3) is a special case, for different boundary conditions (see Refs. [9–11]). In this way we were able to derive explicit expressions for the spectral sum rules of a circular sector and of a symmetric annulus with Dirichlet boundary conditions. The case of boundary conditions allowing a zero mode (which would correspond, for instance, to the circular annulus with Neumann boundary conditions), specifically discussed in Ref. [11], is particularly delicate because of extra contributions that have to be correctly taken into account.

We refer the reader interested in other examples of spectral sum rules for related problems to the references cited in Refs. [9–11].

The purpose of this paper is to extend our previous results to the calculation of sum rules of order one for two dimensional domains: we will show that, by following the adequate prescriptions, it is