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Bi-orthogonal approach to non-Hermitian Hamiltonians with the oscillator spectrum: Generalized coherent states for nonlinear algebras

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A B S T R A C T

A set of Hamiltonians that are not self-adjoint but have the spectrum of the harmonic oscillator is studied. The eigenvectors of these operators and those of their Hermitian conjugates form a bi-orthogonal system that provides a mathematical procedure to satisfy the superposition principle. In this form the non-Hermitian oscillators can be studied in much the same way as in the Hermitian approaches. Two different nonlinear algebras generated by properly constructed ladder operators are found and the corresponding generalized coherent states are obtained. The non-Hermitian oscillators can be steered to the conventional one by the appropriate selection of parameters. In such limit, the generators of the nonlinear algebras converge to generalized ladder operators that would represent either intensity-dependent interactions or multi-photon processes if the oscillator is associated with single mode photon fields in nonlinear media.

1. Introduction

In ordinary quantum mechanics the dynamical variables $\mathcal{O}$ that are susceptible to measurement are called *observables*. These are usually represented by self-adjoint operators $O = O^{\dagger}$ whose eigenvectors form a complete set (i.e., the operators O are *Hermitian*). The latter means that the superposition principle holds. In turn, the reason to restrict O to be self-adjoint is merely practical

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since it is the simplest form to associate its eigenvalues with the result of a measurement of $\mathcal{O}$, which "must always give a real number as result" [1]. However, the reality of the results of any measurement does not imply that the related observables must be represented by self-adjoint operators. Actually, there is a wide class of operators that have real spectrum although they are not self-adjoint. Notable examples are the $\mathcal{PT}$-symmetric Hamiltonians [2,3], the pseudo-Hermitian Hamiltonians [4,5] and the non-Hermitian Hamiltonians generated by supersymmetry, see e.g. [6–13]. Such operators could also represent observables. In practice, they are useful for modeling systems whose phenomenology cannot be explained in terms of the conventional Hermitian approach [14,15]. Some applications include the propagation of light in media with complex-valued refractive index [16], transition probabilities in multi-photon processes [17] and the refinement of diverse techniques of measurement [18].

Recently, we have introduced a class of one-dimensional Hamiltonians H_λ whose eigenvalues are all real although the related potentials V_λ are complex-valued functions [13]. That is, the operators H_λ are not self-adjoint so that their eigenvectors are not mutually orthogonal. This implies that the Sturm–Liouville theory [19,20], which is useful to analyze the completeness of the eigenvectors of any self-adjoint operator, does not apply in the study of H_λ. Thus, it is difficult to determine whether any H_λ satisfies the conditions for representing an observable or not. We may start by assuming that the measurement of a given dynamical variable will give as result any of the eigenvalues of H_λ. The main problem is to find a way to satisfy the superposition principle since the eigenvectors of H_λ are not necessarily complete. Nevertheless, we have shown [21] that the real and imaginary parts of the eigenfunctions of H_λ (i.e., the eigenvectors of H_λ in position-representation) satisfy interlacing theorems that are very similar to those of the Hermitian approaches. The latter is a clear evidence that the eigenfunctions of H_λ might be complete and is consistent with the results reported in [22] for $\mathcal{PT}$-symmetric Hamiltonians. However, the reality of the spectrum of H_λ does not require the invariance under $\mathcal{PT}$ transformations since H_λ is the result of applying the appropriate Darboux transformation on a given Hermitian Hamiltonian H. The main point is that the related complex-valued potentials V_λ satisfy a condition of *zero total area* (the imaginary part of V_λ is continuous in $\mathbb{R}$ and its integral over all the real line is equal to zero), which includes the $\mathcal{PT}$-symmetry as a particular case [21].

The process of adding quantum states to give new quantum states is connected with a mathematical procedure that is always permissible [1]. This demands indeterminacy in the results of observations and was recognized as a breaking point from the classical ideas since the dawn of quantum theory [23]. To ensure that such a fundamental principle is satisfied by the states associated to H_λ, in this work we extend the orthonormal relation obeyed by the eigenvectors of H to an orthonormal property which follows from the simultaneous consideration of the eigenvectors of H_λ and those of $H_\lambda^\dagger$. The approach provides a mathematical structure for working with H_λ and $H_\lambda^\dagger$ as if they were two different faces of the same Hermitian Hamiltonian. Thus, the entire set of eigenvectors of H_λ and $H_\lambda^\dagger$ form a *bi-orthogonal system* so that closure relations can be introduced to accomplish the superposition principle. Moreover, the algebraic properties of the operators that act on the eigenvectors of the non-Hermitian Hamiltonians are easily identified.

The physical model discussed in the present work is represented by a family of non-Hermitian operators H_λ whose spectrum includes all the energies of the harmonic oscillator E_n, $n \geq 0$, plus an additional real eigenvalue $\epsilon < E_0$. Remarkably, the Hermitian oscillator-like systems reported in e.g. [24,25], as well as the conventional oscillator, are recovered as particular cases from our results. The non-Hermitian oscillators are constructed as Darboux transformations of the conventional oscillator and admit two different kinds of ladder operators which give rise to two different families of generalized coherent states. In this context we would like to emphasize that a coherent state is essentially a superposition of the energy eigenvectors of the harmonic oscillator [26,27] (see also [28]). For systems other than the oscillator, the so-called generalized coherent states are concrete superpositions of the eigenvectors of a given observable [29,30]. Thus, to obtain superpositions with properties like those of the coherent states, it is useful to have at hand a complete set of eigenvectors. As the latter is not necessarily the case for non-Hermitian Hamiltonians, the construction of coherent states is a big challenge for such a class of operators in general. However, as we are going to see, the bi-orthogonal approach introduced in this work permits the derivation of such states in simple form. More specifically, we are going to deal with generalized coherent states that are bi-orthogonal superpositions of the energy eigenvectors of non-Hermitian oscillators. Within an approach that