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Quantitative coherence witness for finite dimensional states

Huizhong Ren, Anni Lin, Siying He, Xueyuan Hu *

School of Information Science and Engineering, Shandong University, Jinan 250100, China

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We define the stringent coherence witness as an observable whose mean value vanishes for all incoherent states but nonzero for some coherent states. Such witnesses are proved to exist for any finite-dimension states. Not only is the witness efficient in testing whether a state is coherent, but also its mean value can quantitatively reveal the amount of coherence. For an unknown state, the modulus of the mean value of a normalized witness provides a tight lower bound to the l_1-norm of coherence. When we have some previous knowledge of a state, the optimal witness which has the maximal mean value is derived. It is proved that for any finite dimension state, the mean value of the optimal witness, which we call the witnessed coherence, equals the l_1-norm of coherence. In the case that the witness is fixed and the incoherent operations are allowed, the maximal mean value can reach the witnessed coherence if and only if certain relations between the fixed witness and the initial state are satisfied. Our results provide a way to directly measure the coherence in arbitrary finite dimension states and an operational interpretation of the l_1-norm of coherence.

1. Introduction

Instead of state tomography, the entanglement can be tested by measuring only one observable, called entanglement witness [1]: if the mean value is negative, the state must be entangled. Moreover, if the mean value is well below zero, one can also infer that the entanglement is very large [2,3]. Various quantum games are designed on the basis of entanglement witness, indicating that all entangled states can be used as a resource in these games [4–8]. Nevertheless, using entanglement

* Corresponding author.
E-mail address: xyhu@sdu.edu.cn (X. Hu).

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witness to test the entanglement remains challenging both experimentally and theoretically, partly because the optimization problem is hard [9,10].

Quantum coherence [11,12], a fundamental property in quantum theory, is closely related to the resource theory of quantum entanglement [13–17]. On the prefixed incoherent basis $\{|j\rangle\}$, the incoherent states [18] are defined as those with diagonal density matrices $\mathcal{I} := \{\rho_I : \rho_I = \sum_j p_j |j\rangle\langle j|\}$, and the incoherent operations [18] are those with incoherent Kraus decompositions $\mathcal{IO} := \{\Lambda^I : \Lambda^I(\cdot) = \sum_n K_n(\cdot)K_n^\dagger,\ \text{s.t.}\ K_n \mathcal{I} K_n^\dagger \subset \mathcal{I}\}$. Different coherence measures have been proposed [18–22] and their monotonicity under incoherent operations has been analyzed [23–25]. Among those measures, the l_1-norm of coherence has ideal properties, such as strong monotonicity [18] and computational simplicity, but lacks an operational interpretation.

Inspired by the entanglement witness, the coherence witness was proposed and proved to be related to the randomness of coherence and l_1-norm of coherence for certain classes of states [21,26–28]. Similar to entanglement witness, the coherence witness was defined as an observable whose mean value is nonnegative for incoherent states and hence a negative mean value indicates the coherence. A recent experiment [29] measures the mean value of the optimal witness, and find that for a class of single-qubit states, its opposite coincides with the robustness of coherence as well as the l_1-norm of coherence.

In this paper, we propose a more stringent coherence witness for any finite-dimensional coherent state, and solve several optimization problems. Different from the traditional witnesses, our coherence witness has zero mean value for all of the incoherent states, and hence a nonzero mean value, no matter positive or negative, indicates the coherence. This witness is efficient, in the sense that most of coherent states can be detected by only one witness, and the number of unsure states reduces fast as the number of witnesses increases.

Another advantage of our witness is that it simplifies the optimization problems. For any given normalized coherence witness W and its mean value c, we prove that the coherence C_{l_1} in the measured state is at least $|c|$. When a state ρ is known, we derive the explicit form of the optimal witness (which is a normalized witness with maximum mean value in ρ), and prove that the mean value of optimal witness is just the l_1-norm of coherence in ρ. This connection builds an operational interpretation of C_{l_1}. Intuitively, the measurement of some witnesses is very difficult due to the experimental constraints, so it is important to study the following problem: if we fix the witness W, can we optimize the state using incoherent operations such that the mean value of W can reach that of the optimal witness? We give a positive answer when a qubit state is considered, but a negative one in high-dimension case. Based on this result, we design a quantum game where quantum coherence is a resource but some amount of coherence cannot be activated in this game.

2. Existence of coherence witness

Entanglement witness is proved to exist for any entangled states based on Hahn–Banach Theorem and the convexity of the separable states [30]. Following the similar idea, the coherence witness was recently proposed [21]. For any state $\rho \notin \mathcal{I}$, there is a Hermit operator $\tilde{W}$ such that $\mathrm{tr}(\tilde{W}\rho) < 0$ but $\mathrm{tr}(\tilde{W}\rho_I) \geq 0$ for all incoherent state $\rho_I \in \mathcal{I}$. Hence $\tilde{W}$ is called a coherence witness, namely, detecting a negative mean value of $\tilde{W}$ reveals the quantum coherence in ρ.

When single-qubit states are considered, the Bloch representation provides a geometric picture for the coherence witness. Let $\boldsymbol{\sigma} := (\sigma_x, \sigma_y, \sigma_z)$ be Pauli matrices. Any single-qubit state $\rho = \frac{1}{2}(\mathbb{I} + \boldsymbol{b}\cdot\boldsymbol{\sigma})$ is presented as a three-dimension real vector $\boldsymbol{b}$ with norm $b \leq 1$. In the incoherent basis, all of the incoherent states are on the z-axis. For a Hermit operator $\tilde{W} = w_0\mathbb{I} + \boldsymbol{w}\cdot\boldsymbol{\sigma}$, the states satisfying $\mathrm{tr}(\tilde{W}\rho) \equiv w_0 + \boldsymbol{w}\cdot\boldsymbol{b} = 0$ consist of a plane in the Bloch space. When $\tilde{W}$ is a coherence witness, all of the states on the z-axis are above or on the plane, and the coherence in the states below can be witnessed by $\tilde{W}$. In order to witness more coherent states, we should push the plane $\mathrm{tr}(\tilde{W}\rho) = 0$ closer to the z-axis. The best we can do is that the z-axis is just on the plane. This leads to an interesting result that $\mathrm{tr}(\tilde{W}\rho_I) = 0,\ \forall \rho_I \in \mathcal{I}$, so $\mathrm{tr}(\tilde{W}\rho) \neq 0$ indicates the existence of quantum coherence in ρ.

Inspired by the above observation, we propose the stringent coherence witness and prove its feasibility for any finite-dimension states.