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A difference-equation formalism for the nodal domains of separable billiards

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ABSTRACT

Recently, the nodal domain counts of planar, integrable billiards with Dirichlet boundary conditions were shown to satisfy certain difference equations in Samajdar and Jain (2014). The exact solutions of these equations give the number of domains explicitly. For complete generality, we demonstrate this novel formulation for three additional separable systems and thus extend the statement to all integrable billiards.

1. Introduction

The classical billiard is a dynamical system consisting of a point particle moving freely in an enclosure, alternating between motion along a straight line and elastic reflections off the boundary, dictated by Snell's law [1–4]. This sequence of specular reflections is captured by the billiard map, which completely describes the motion of the particle. These simple systems exhibit a wide range of dynamical behavior from order to chaos [5–7], depending on the shape of their boundary. Classically speaking, an integrable billiard is defined to be one in which the number of constants of motion equals the number of degrees of freedom. A long-standing conjecture by Birkhoff states that among all billiards inside smooth convex curves, ellipses are characterized by integrability of the billiard map [8]. On the other hand, examples of their ergodic counterparts (for example, Bunimovich stadium [9], dispersive Sinai billiards [10]) are equally well-known [11].

Over the last two decades, the quantum analogues of these systems, quantum billiards, have been experimentally realized in gated, mesoscopic GaAs tables [12], microwave cavities [13] and

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ultracold atom traps [14,15]. The eigenfunctions of these planar billiards organize themselves into regions, or domains, with positive and negative signs, often in remarkably complicated geometric shapes. Formally, such nodal domains may be defined as the maximally connected regions wherein the wavefunction does not change sign. Academic interest in various statistical measures pertaining to the number of nodal domains, ν, was piqued [16] in light of discoveries such as a new criterion for chaos in quantum mechanics [17] and the presence of geometric information about the system in its nodal count sequences [18]. Experimentally, nodal domains have also been the focus of much attention as documented in Refs. [19–21]. Unfortunately, quantifying the nodal patterns is a major challenge since it is extremely hard to discern any order when ν is arranged in an ascending order of energy.

In principle, the problem seems (deceptively) straightforward—for each billiard of interest, we need only solve the Schrödinger equation in appropriate coordinates, and count the domains as a function of the two quantum numbers m and n of the system. However, in order to arrive at a functional form for ν, it was recently discovered [22] that it is more fruitful to analyze the differences $\Delta_{kn}\,\nu(m,n) = \nu_{m+kn,n} - \nu_{m,n},\ k \in \mathbb{N}$, instead. The proposition put forth therein was that for any integrable billiard in two dimensions, "one of $\Delta_{kn}\nu\,(m,n) = \Phi(n)$ and $\Delta^2_{kn}\nu\,(m,n) = \Phi(n)\ \forall\, m,n$ holds for some $\Phi\ :\ \mathbb{R}\ \rightarrow\ \mathbb{R}$, which is determined only by the geometry of the billiard". This difference-equation formulation, which proves to be of great practical utility in determining ν analytically (especially in the study of non-separable polygons, cf. [23]), was illustrated in [22] for a few separable and all non-separable, integrable polygonal shapes. The natural question that one then asks is: can *all* integrable, planar billiards be similarly characterized by a difference equation in their nodal domain counts? The answer, as we show, is in affirmative.

For definiteness, we demonstrate that the statement can be generalized to *all* planar, integrable billiards, which includes both convex billiards, with smooth boundaries, and billiards in polygons. Since the two-dimensional Helmholtz equation is separable in only four coordinate systems – the Cartesian, polar, elliptic and parabolic coordinates [24,25] – the additional geometries which must be considered are the elliptical billiard, the system of two confocal parabolas and various annular regions of the above. For all such separable systems, $k = 1$ and this may be regarded as a fingerprint of separability in the difference equations themselves.

2. Circular annuli and sectors

The circle is but a special case within the class of elliptical billiards and is described in two-dimensional polar coordinates by $\mathcal{D} = \{(r,\theta) : 0 \le r \le R,\ 0 \le \theta \le 2\pi\}$. The Helmholtz equation for this system, $r\,\partial_r(r\,\partial_r\psi) + \partial^2_\theta\,\psi + k^2r^2\psi = 0$, can be separated into radial and angular components, where the radial solution is a cylindrical Bessel function of the first kind, denoted by $J_m(kr)$, m being the angular quantum number, and the angular function is simply $\exp(\mathrm{i}\,m\,\theta)$ [26,27]. In Ref. [22], it was shown that for the circular billiard, $\Delta_n\nu(m,n) = \nu_{m+n,n} - \nu_{m,n} = 2n^2$, which gives $\nu_{m,n} = 2\,mn$. Here, we study the annular regions of this billiard where the domain is restricted, first in the radial variable and then, in both the radial and angular variables. For the annulus where $\theta \in [0, 2\pi]$, the quantum number m so obtained is integral and thus the general solution to the radial Helmholtz equation is a linear combination of the mth-order Bessel functions of the first and second kind, $J_m(kr)$ and $Y_m(kr)$. Assuming that the annular region is enclosed within the radii r_1 and r_2, the boundary conditions $\psi = 0$ for $r = r_1, r_2$ suggest the form

$$\psi_{m,n}(r,\theta) = A\left[J_m(k_nr)\,Y_m(k_nr_1) - J_m(k_nr_1)\,Y_m(k_nr)\right]\cos(m\pi\theta), \tag{2.1}$$

where A is a normalization constant and k_n is the value of k such that r_2 is the nth zero of the radial solution in the domain of its definition (excluding the zero at r_1). Adopting these conventions for m and n, the recurrence relation satisfied by the nodal domains is $\nu_{m+n,n} - \nu_{m,n} = 2n^2$, and on scrutinizing individual cases, the formula $\nu_{m,n} = 2\,mn$ is obtained. These results are identical to those obtained for the full circle and the corresponding examples have been illustrated in Fig. 2.1(a). On the other hand, for regions where the domain is restricted in the radial as well as the angular variable, the Dirichlet boundary conditions become analogous to those imposed on a rectangular billiard, which can be argued as follows. In the cases where θ runs from 0 to 2π, the requirement of periodicity in θ