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# Classical impurities and boundary Majorana zero modes in quantum chains

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ABSTRACT

We study the response of classical impurities in quantum Ising chains. The $\mathbb{Z}_2$ degeneracy they entail renders the existence of two decoupled Majorana modes at zero energy, an exact property of a finite system at arbitrary values of its bulk parameters. We trace the evolution of these modes across the transition from the disordered phase to the ordered one and analyze the concomitant qualitative changes of local magnetic properties of an isolated impurity. In the disordered phase, the two ground states differ only close to the impurity, and they are related by the action of an explicitly constructed quasi-local operator. In this phase the local transverse spin susceptibility follows a Curie law. The critical response of a boundary impurity is logarithmically divergent and maps to the two-channel Kondo problem, while it saturates for critical bulk impurities, as well as in the ordered phase. The results for the Ising chain translate to the related problem of a resonant level coupled to a 1d p-wave superconductor or a Peierls chain, whereby the magnetic order is mapped to topological order. We find that the topological phase always exhibits a continuous impurity response to local fields as a result of the level repulsion of local levels from the boundary Majorana zero mode. In contrast, the disordered phase generically features a discontinuous

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magnetization or charging response. This difference constitutes a general thermodynamic fingerprint of topological order in phases with a bulk gap.



## 1. Introduction

In recent years there was a substantial boost in the search for signatures of Majorana zero modes (MZM) that may emerge as localized quasiparticles in various condensed matter realizations because of their potential for quantum computation [1,2]. MZMs occurring at boundaries or defects (domain walls and vortices) in low-dimensional topological superconductors are of particular importance because of their non-Abelian anyonic statistics that uncovers new prospects for storage and manipulation of quantum information [3,4].

In his seminal paper Kitaev [5] proposed a one-dimensional model of a spinless *p*-wave superconductor (1DPS)

$$H = -\mu \sum_{n=1}^{N} (a_n^\dagger a_n - 1/2) + \frac{1}{2} \sum_{n=1}^{N-1} \left( t a_n^\dagger a_{n+1} + \Delta a_n^\dagger a_{n+1}^\dagger + \text{h.c.} \right) \tag{1}$$

(the usual negative sign of the hopping term can be obtained by the transformation $a_n \to (-1)^n a_n$, which changes the signs of $t$ and $\Delta$). This model has a topologically non-trivial massive phase that supports localized Majorana modes at the ends of the chain. For a macroscopically large system these boundary modes can be regarded as unpaired, in which case they represent a non-local realization of a doubly degenerate fermionic zero-energy state. The spatial separation of the two MZMs ensures the immunity of the topologically degenerate ground state of the 1DPS against weak local perturbations (as long as quasi-particle poisoning can be neglected, and thus fermion parity is conserved), making such a system potentially useful for the needs of quantum computation. Thus it is of great theoretical interest and practical importance to identify the physical properties of the edge of such a 1D system that can serve as evidence for the existence of boundary MZMs.

It has soon been realized that principal features of the Kitaev 1D model [5] can be reproduced experimentally using a quantum wire with a strong spin–orbit coupling in the presence of an external magnetic field and the proximity effect with a conventional s-wave superconducting substrate [6,7]. Much theoretical and experimental effort is currently going into finding unambiguous signatures of MZMs in various set-ups. Important steps forward in this direction include tunneling spectroscopy experiments [8,9], whose findings, in particular, the zero-bias conductance peak observed in one-dimensional semiconductor–superconductor contacts, were consistent with theoretical predictions (see Ref. [1] for a recent review).

Closely related to the 1DPS model is the quantum Ising chain (QIC), described by the Hamiltonian:

$$H = -J \sum_{n=1}^{N-1} \sigma_n^x \sigma_{n+1}^x - h \sum_{n=1}^{N} \sigma_n^z . \tag{2}$$

Here $\sigma_n^\alpha$ are Pauli matrices, $J > 0$ is the exchange interaction and $h$ is a transverse magnetic field which endows the spins with quantum dynamics. The model possesses a $\mathbb{Z}_2$-symmetry associated with the global transformation $P_S \sigma_n^x P_S^{-1} = -\sigma_n^x$, where $P_S = \prod_{j=1}^{N} \sigma_j^z$, $[H, P_S] = 0$. This is an exactly solvable quantum 1D model which, by virtue of the transfer matrix formalism, is related to the classical 2D Ising model [10–12]. The Jordan–Wigner (JW) transformation maps the many-body problem (2) onto a quadratic model of spinless fermions, the latter actually being a particular realization of the 1DPS (1) with a fine-tuned pairing amplitude $\Delta = \pm t$. Close to criticality, in the field-theoretical limit, the QIC represents a $(1+1)$-dimensional theory of a massive Majorana