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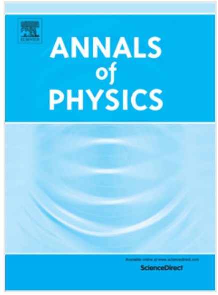

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Similarity solutions of Reaction-Diffusion equation with space- and time-dependent diffusion and reaction terms

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We consider solvability of the generalized reaction-diffusion equation with both space- and time-dependent diffusion and reaction terms by means of the similarity method. By introducing the similarity variable, the reaction-diffusion equation is reduced to an ordinary differential equation. Matching the resulting ordinary differential equation with known exactly solvable equations, one can obtain corresponding exactly solvable reaction-diffusion systems. Several representative examples of exactly solvable reaction-diffusion equations are presented.

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I. INTRODUCTION

Many natural phenomena involve the change of concentration/population of one or more substances/species distributed in space under the influence of two processes: local reaction which modify the concentration/population, and diffusion which causes the substances/species to spread in space. Such phenomena are well modelled by the reaction-diffusion equation (RDE).

The general form of RDE of the concentration $W(x,t)$ of a single component in one spatial dimension is

$$\frac{\partial}{\partial t}W(x,t) = D\frac{\partial^2}{\partial x^2}W(x,t) + f(W), \tag{1}$$

where D is the constant diffusion coefficient and $f(W)$ is the reaction term which accounts for the local reaction. Eq. (1) is also called the KPP (Kolmogorov-Petrovsky-Piscunov) equation, named after the authors who first studied some of the mathematical properties of the RDE. This equation encompasses the diffusion (heat) equation ($f=0$) and the Fokker-Planck equation (when f is a gradient term of some function linear in W) [2].

Different forms of the reaction term f have been proposed to describe different phenomena. For instance, the choice $f = W(1-W)$ yields the Fisher equation employed in the study of wave propagation of advantageous genes in a population [3] and evolution of a neutron population in a nuclear reactor [4]. Rayleigh-Bernard convection is studied using RDE with $f = W(1-W^2)$ [5], while combustion and shock waves phenomena invoke RDE with $f = W(1-W)(W-a)(0<a<1)$ [6]. Generalization and extension of the KPP equation to higher dimensions and multi-component cases also find interesting applications in chemical kinetics [7], pattern formation and morphogenesis [8], nerve pulse propagation in nerve systems [9], and other biological systems [10].

In view of its broad applicability, it is thus desirable to obtain analytic solutions of the RDE for as many systems as possible. However, just as any equation in sciences, solving the RDE exactly is in general a formidable task, except in a few simplified cases. Fortunately, for many cases of the RDE mentioned before, exact solutions can be found in the form of travelling wave solutions [11].

In this paper we would like to consider exact solvability of the RDE in terms of the similarity solutions [12]. This is motivated by our recent works on similarity solutions of the Fokker-Planck equation, which as mentioned before is a subclass of the RDE [13–15]. We found it very interesting that for a class of the Fokker-Planck equation with time- and space-dependent coefficients, a general formula of exact solutions can be obtained in closed form by the similarity method, both for fixed and moving boundaries. One advantage of the similarity method is that it allows one to reduce the partial differential equation under consideration to an ordinary differential equation which is generally easier to solve, provided that the original equation possesses proper scaling property under certain scaling transformation of the basic variables. Here we would like to extend our previous consideration to the RDE. It turns out that, similar to the Fokker-Planck case, one can determine the possible functional forms of the diffusion and the reaction term of the RDE in order to get exactly solvable system with similarity solutions.

This paper is organized as follows. Sect. II discusses the scaling properties of the RDE. Sect. III introduces the corresponding similarity variable and scaling forms of the relevant functions, which are used to reduce the RDE into an ordinary differential equation. The equation of continuity is discussed in Sect. IV which helps to identify two types of scaling behaviours of the RDE. Some examples of these two types of scaling reaction-diffusion (RD) systems are presented in Sect. V and VI, respectively. Sect. VII concludes the paper.