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Propagators of isochronous an-harmonic oscillators and Mehler formula for the exceptional Hermite polynomials

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ABSTRACT

It is shown that fundamental solutions $K^{\sigma}(x, y; t) = \langle x|e^{-iH^{\sigma}t}|y\rangle$ of the non-stationary Schrödinger equation (Green functions, or propagators) for the rational extensions of the Harmonic oscillator $H^{\sigma} = H_{osc} + \Delta V^{\sigma}$ are expressed in terms of elementary functions only. An algorithm to calculate explicitly K^{σ} for an arbitrary increasing sequence of positive integers σ is given, and compact expressions for $K^{\{1,2\}}$ and $K^{\{2,3\}}$ are presented. A generalization of Mehler's formula to the case of exceptional Hermite polynomials is given.

1. Introduction

Propagator, or Green function of the non-stationary Schrödinger equation, completely describes quantum dynamics in the Feynman path-integral formulation of quantum mechanics [1]. In this work we present new explicit examples of propagators in the case of one-dimensional Schrödinger equation. Namely, we study a quantum particle moving in the potential of a rationally extended Harmonic oscillator [2]. In this case the evolution of wave packets is periodic due to the quasi-equidistant spectrum of the Hamiltonian [3]. Therefore such deformations of the Harmonic oscillator are known as isochronous anharmonic oscillators [4]. It was proved in [5] that all monodromy free rational extensions of the Harmonic oscillator can be obtained by a finite chain of Darboux transformations. In other words, each rational extension V^{σ} of the Harmonic oscillator is defined by a sequence of

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levels $\sigma = \{\sigma[[1]], \sigma[[2]], \ldots, \sigma[[-1]]\}$, $\sigma[[i]] \in \mathbb{N}$ which are deleted from the spectrum by Darboux–Crum transformations [2].

The simplest rational extension is given by the potential [6]

$$V^{\{1,2\}}[x] = \frac{x^2}{4} + 2\left(1 + 2\frac{(x^2-1)}{(x^2+1)^2}\right), \tag{1}$$

which leads to the quasi-equidistant spectrum for the Hamiltonian [3], $E_n = n + \frac{1}{2}$, where $n \in \mathbb{N}_0 \setminus \{1, 2\}$.

Another example is the two-well perturbation of the oscillator

$$V^{\{2,3\}}[x] = \frac{x^2}{4} + 2\left(1 + 4x^2\frac{x^4-9}{(x^4+3)^2}\right) \tag{2}$$

with the quasi-equidistant spectrum, $E_n = n + \frac{1}{2}, n \in \mathbb{N}_0 \setminus \{2, 3\}$.

Darboux transformations represent a powerful tool to manipulate physical properties of one-dimensional quantum systems, [6] and to construct (polynomial) supersymmetric extensions of quantum mechanics (SUSY QM) [7,8]. Exactly solvable models obtained by Darboux transformations are widely applied in nuclear physics, condensed matter physics, quantum optics, etc. [9–14].

The possibility to calculate propagators using ideas of supersymmetric quantum mechanics and the theory of solitons was considered in [15]. Quasiclassical approach to propagators and path integration in SUSY QM was developed in [16]. In the case of shape-invariant potentials, SUSY relations between propagators allow to calculate them explicitly [17,18]. We proposed a more general approach to calculations of propagators in SUSY QM without restricting ourselves by shape-invariant potentials in [19–22]. This approach can be extended and applied to the analysis of quantum tunnelling in multi-well potentials [23,24]. In the case of generalized Schrödinger equation the SUSY propagators were calculated in [25]. Green functions of the Dirac equation were studied by means of SUSY QM in [26].

Here we re-examine results of [22], where propagators $K^{\{k,k+1\}}$ for the $V^{\{k,k+1\}}$ family of potentials were defined by means of a generating function $S(x, y; t\lfloor J)$ which contains the error-function. We will extend this result to arbitrary sequences σ. Moreover, we will show that propagators K^σ are expressed by elementary functions only.

In the case of potentials (1) and (2) we will obtain the following propagators

$$K^{\{1,2\}}(x, y; t) = \mathrm{e}^{-2\mathrm{i}t} K_{\mathrm{osc}}(x, y; t)\left(1 - \frac{4\mathrm{i}\,\sin t\left[xy - \mathrm{e}^{\mathrm{i}t}\right]}{(1+x^2)(1+y^2)}\right), \tag{3}$$

$$\begin{aligned} K^{\{2,3\}}(x, y; t) &= \mathrm{e}^{-2\mathrm{i}t} K_{\mathrm{osc}}(x, y; t) \\ &\times \left(1 - \frac{8\mathrm{i}\,\sin t\left[xy(x^2y^2-3) - 3(x^2+y^2)\cos t - 3\mathrm{i}\,(x^2y^2+1)\sin t\right]}{(3+x^4)(3+y^4)}\right), \end{aligned} \tag{4}$$

where the propagator of the Harmonic oscillator [1] is used

$$K_{\mathrm{osc}}(x, y, t) = \frac{1}{\sqrt{4\pi\mathrm{i}\,\sin t}}\,\mathrm{e}^{\frac{\mathrm{i}\,[(x^2+y^2)\cos t - 2xy]}{4\sin t}}. \tag{5}$$

The paper is organized as follows. In the first section, we recall how to construct rational extensions of the Harmonic oscillator. In the second section we first give an implicit SUSY-based expression for propagators $K_\sigma(x, y; t)$ in terms of generating function.

Analysing the generating function $S(x, y; t\lfloor J)$ we will define a suitable rational ansatz to compute propagators K^σ. In the general case, propagators for the rationally extended oscillators have the following structure

$$K^\sigma(x, y; t) = K_{\mathrm{osc}}(x, y; t)\frac{\sum_{k=0}^{\sigma[[-1]]+1} Q_k^\sigma(x, y)\mathrm{e}^{-\mathrm{i}kt}}{\sum_{k=0}^{\sigma[[-1]]+1} Q_k^\sigma(x, y)} \tag{6}$$