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Generalized uncertainty principle and self-adjoint operators

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A B S T R A C T

In this work we explore the self-adjointness of the GUP-modified momentum and Hamiltonian operators over different domains. In particular, we utilize the theorem by von-Neumann for symmetric operators in order to determine whether the momentum and Hamiltonian operators are self-adjoint or not, or they have self-adjoint extensions over the given domain. In addition, a simple example of the Hamiltonian operator describing a particle in a box is given. The solutions of the boundary conditions that describe the self-adjoint extensions of the specific Hamiltonian operator are obtained.

1. Introduction

One of the old, vexing, and still unsolved problems of Theoretical Physics is that of merging gravity with quantum field theory. Nowadays, the incarnation of this combination is the several theories of quantum gravity (such as String theory) that we have. One of the new features predicted by these theories is the minimal measurable length (for a recent review see [1], and references there in). This feature led to the generalization of the Heisenberg Uncertainty Principle, i.e. the Generalized Uncertainty Principle (henceforth abbreviated to GUP). In addition, GUP can also be considered as an

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outcome of modifications/corrections to the conventional Heisenberg algebra satisfied by the two canonically conjugate observables: position $\mathbf{x}$ and momentum $\mathbf{p}$. According to String Theory, the conventional Heisenberg algebra gains an extra term which is quadratic in momentum $\mathbf{p}$, in the Planck regime. In [2], the authors used the following modified Heisenberg algebra consistent with String Theory

$$\left[\hat{x}, \hat{p}\right] = i\hbar \left(1 + \beta p^2\right). \tag{1}$$

Black Hole physics, and Doubly Special Relativity (DSR) propose a correction in the Planck regime with the extra term to be linear in momentum $\mathbf{p}$. In [3,4], the authors considered modifications to the conventional Heisenberg algebra, which includes both linear and quadratic terms in momentum, namely

$$[\hat{x}, \hat{p}] = i\hbar \left[1 - 2\alpha p + 4\alpha^2 p^2\right]. \tag{2}$$

Utilizing the following uncertainty relationship satisfied by any two operators $\hat{A}$ and $\hat{B}$

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2} \left\| \left\langle \left[\hat{A}, \hat{B}\right] \right\rangle \right\| \tag{3}$$

where $\Delta\hat{A}$ and $\Delta\hat{B}$ stand for the standard deviations of the corresponding operators, a GUP between position x and momentum p is obtained

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left[1 - 2\alpha \langle p \rangle + 4\alpha^2 \left\langle p^2 \right\rangle\right] \tag{4}$$

where $l_{pl} \approx 10^{-35}$ m is the Planck length, $\alpha = \alpha_0\, l_{pl}/\hbar$ is the GUP parameter, and it is normally assumed that α_0 is of order 1. In [3,4], the authors suggested an upper bound on α_0 by stating that its value cannot exceed 10^{17} which is precisely the electroweak length scale. This prediction comes about due to the fact that if α_0 were to be any larger, such an intermediate length scale would have been observed. Now, it is easy to verify the fact that the above two equations, i.e., Eqs. (2) and (4), predict a minimum uncertainty in position, i.e, Δx_{min}, and a maximum uncertainty in momentum, i.e., Δp_{max},[1]

$$\Delta x_{min} \propto \alpha_0 l_{pl} \tag{5}$$

$$\Delta p_{max} \propto \frac{M_{pl}c}{\alpha_0} \tag{6}$$

where M_{pl} is the Planck Mass. It can be shown that the following representations of the position and momentum operators satisfy the modified Heisenberg algebra given by Eq. (2)

$$\hat{x} = \hat{x}_0 \tag{7}$$

$$\hat{p} = \hat{p}_0 \left(1 - \alpha\hat{p}_0 + 2\alpha^2\hat{p}_0^2\right) \tag{8}$$

with $\hat{x}_0$ and $\hat{p}_0$ satisfying the ordinary canonical commutation relations $\left[\hat{x}_0, \hat{p}_0\right] = i\hbar$. Here, $\hat{p}_0$ can be interpreted as being the total momentum of a particle at low energies and having the standard representation, namely, in one dimension,

$$\hat{p}_0 = -i\hbar\frac{d}{dx}. \tag{9}$$

Now, considering any non-relativistic Hamiltonian of the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\mathbf{r}) \tag{10}$$

[1] It is noteworthy that DSR theories also introduce the features of minimal measurable length and maximum measurable momentum [5,6].