Annals of Physics 360 (2015) 150–160

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Phase transitions in Wick-rotated $\mathcal{PT}$-symmetric optics

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HIGHLIGHTS

- Proposal of optical models of Wick-rotated $\mathcal{PT}$-symmetric systems.
- Physical signatures of spectral phase transitions.
- Dissipative $\mathcal{PT}$-symmetric dimers and phase locking transitions.

ARTICLE INFO

Article history:
Received 23 December 2014
Accepted 4 May 2015
Available online 11 May 2015

Keywords:
$\mathcal{PT}$ symmetry breaking
Dynamic laser instability
Coupled oscillators

ABSTRACT

$\mathcal{PT}$-symmetric models with a Wick rotation of time ($t \to \pm it$) show spectral phase transitions that are similar to those of dissipative systems driven out of equilibrium. Optics can provide an accessible test bed to explore spectral phase transitions of Wick-rotated $\mathcal{PT}$-symmetric models. This is shown by considering the transverse dynamics of laser light in optical cavities with variable reflectivity and tilted mirrors. Two specific examples are discussed: the optical analogue of the hydrodynamic Squire model of vorticity, and the Wick-rotated $\mathcal{PT}$-symmetric nonlinear dimer model. In the latter case the spectral phase transition is associated with the universal phase locking–unlocking transition in Adler's theory of coupled oscillators.

1. Introduction

In 1998, Bender and Boettcher showed that a wide class of non-Hermitian Hamiltonians $\hat{H}$ can possess entirely real spectra as long as they respect parity-time ($\mathcal{PT}$) symmetry [1,2]. While the

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http://dx.doi.org/10.1016/j.aop.2015.05.008

implications of $\mathcal{PT}$ symmetry in theoretical physics are still a matter of debate [3], classical systems such as optical [4–10] and electronic [11] systems provide an accessible test bed where the $\mathcal{PT}$ symmetry notion can be explored. $\mathcal{PT}$-symmetric Hamiltonians show a sharp spectral transition when a control parameter is varied, with the appearance of pairs of complex conjugate energies in the broken $\mathcal{PT}$ phase. The phase transition is associated with the appearance of exceptional points [12] or spectral singularities [13,14]. Similar transitions are found in pseudo-Hermitian Hamiltonians [15], pseudo-$\mathcal{PT}$-symmetric driven Hamiltonians [16], Wick-rotated $\mathcal{PT}$ symmetric Hamiltonians [17], and in Liouvillean operators in the Lindblad form [18]. In particular, in Ref. [18] it was shown that a combination of unitary and antiunitary symmetry of quantum Liouvilleans associated to certain open quantum systems implies a dihedral symmetry of the complex Liouvillean spectrum. A different (and not necessarily dihedral) symmetry of the spectrum is found in Hamiltonian systems after application of Wick rotation [19], which consists in rotating the time axis by $\pm\pi/2$ in complex plane, i.e. to the transformation $t \rightarrow \pm it$. In particular, Wick rotation of $\mathcal{PT}$ symmetric Hamiltonians generates the same spectral transition in complex plane, but rotated by $\pm\pi/2$. Complexification of time and corresponding rotation of the spectrum deeply changes the physical signatures of the symmetry breaking. Physical systems described by Wick-rotated $\mathcal{PT}$-symmetric models are found in hydrodynamics [17,20] and in certain gauge field theories [21]. A paradigmatic example is the Squire model, which was introduced in hydrodynamics to describe the normal vorticity of a plane Couette flow [17,20,22] and to explain large transient growths of perturbations in spite of the linear stability of the underlying flow [20,23,24].

In this work we show that spectral phase transitions in Wick-rotated $\mathcal{PT}$-symmetric Hamiltonians behave like phase transitions in dissipative systems driven out of equilibrium, and that optics can provide an accessible test bed where such spectral phase transitions and their physical signatures can be explored. We consider transverse laser dynamics in optical resonators with variable reflectivity mirrors [25] and discuss, as examples, the optical realization of the hydrodynamic Squire model [20,22] and the Wick-rotated nonlinear $\mathcal{PT}$-symmetric dimer model [26,27].

2. Optical resonator model of Wick-rotated $\mathcal{PT}$-symmetric systems

A dynamical system that realizes a Wick-rotated $\mathcal{PT}$-symmetric model is described quite generally by the following equation for an order parameter $\psi(x,t)$

$$\partial_t\psi = -\hat{H}\psi - |\psi|^2\psi \tag{1}$$

where $\hat{H}$ is the $\mathcal{PT}$-symmetric operator defined by

$$\hat{H} = -\partial_x^2 + V(x) - g_0. \tag{2}$$

In Eq. (2), $V(-x) = V^*(x)$ is the $\mathcal{PT}$-symmetric complex potential, whereas g_0 is a constant real parameter that just provides a shift of the real part of the energies of $\hat{H}$. In Eq. (1), a cubic nonlinear term is added to limit the growth of unstable modes of $\hat{H}$. In optics, a possible realization of Eq. (1) is provided by transverse laser dynamics in an optical resonator with variable reflectivity and aspherical mirrors. Let us consider the optical cavity shown in Fig. 1 with a one spatial transverse coordinate X. The resonator comprises two end mirrors in a nearly self-imaging configuration [28], one totally-reflective flat mirror (mirror 2) and the other one a variable-reflectivity and aspherical mirror (mirror 1). The reflectivity of mirror 1 is given by $r(X) = \sqrt{R(X)}\exp[i\Delta(X)]$, where $R(X)$ and $\Delta(X)$ are the transversely-varying power reflectance and phase shift introduced by the aspherical surface of the mirror. The gain medium, placed close to mirror 1, is assumed to have a fast polarization and population relaxation rates (class-A laser [28,29]; e.g. He–Ne, Ar^+, Kr^+ or dye lasers). A set of two focusing lens of focal length f and a Gaussian aperture, placed in the focal plane of the lenses, provide spectral filtering of the optical field at the plane γ in the cavity [28]; see Fig. 1. Indicating by $t(X) = \exp(-X^2/w_a^2)$ the spectral transmission of the Gaussian aperture of size w_a and neglecting diffraction (propagative) effects in the gain medium, assuming single-longitudinal mode oscillation the evolution of the electric field envelope $E(X,T)$ at plane γ in the cavity is governed by the following equation (see Appendix A for details)

$$T_R\partial_T E = \mathcal{D}\partial_X^2 E - V(X)E + gE \tag{3}$$