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Fields, particles and universality in two dimensions

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A R T I C L E   I N F O

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A B S T R A C T

We discuss the use of field theory for the exact determination of universal properties in two-dimensional statistical mechanics. After a compact derivation of critical exponents of main universality classes, we turn to the off-critical case, considering systems both on the whole plane and in presence of boundaries. The topics we discuss include magnetism, percolation, phase separation, interfaces, wetting.

1. Introduction

Statistical systems close to a second order phase transition point exhibit properties which do not depend on details of the microscopic interaction but only on global features such as internal symmetries and space dimensionality. These properties are called universal, and systems sharing them are said to belong to the same universality class. Renormalization theory has progressively clarified how the divergence of the correlation length as the critical point is approached allows for the emergence of universality, and how field theory actually is the theory of universality classes.

While these ideas apply in any dimension $d \geq 2$, to the point that d serves as an expansion parameter in the Wilson–Fisher approach to renormalization, different dimensionalities exhibit specific features which are intrinsically non-perturbative and determine distinctive qualitative properties of universal behavior. The two-dimensional case, in particular, is characterized by features such as absence of spontaneous breaking of continuous symmetries, fermionization (or, conversely, bosonization), correspondence between interfaces and particle trajectories, infinite dimensional

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conformal symmetry at criticality. The last property is in turn at the origin of a further, remarkable peculiarity of the two-dimensional case, i.e. the existence of exact solutions, both at criticality and near criticality, and for all universality classes.

The aim of this article is to illustrate how field theory leads to the exact description of universality classes of classical equilibrium statistical mechanics in two dimensions. The presentation focuses on basic ideas and examples, with a minimal amount of technical aspects. It relies on several recent results, and is original for the spectrum of the topics discussed, ranging from percolation to wetting, and the way they are cast into a unified theoretical framework.

The discussion is structured according to the following path. In the next section we recall generalities about universality, field theory and its particle description. In Section 3 we give a concise overview of two-dimensional conformal field theory suitable for our subsequent discussion; in particular, we do not focus on minimal models, leaving room for cases, like percolation, in which the bulk correlation functions of the order parameter do not satisfy known differential equations. In Section 4 we show how the critical indices of main universality classes, including percolation and self-avoiding walks, can be identified directly in field theory exploiting the insight coming from the particle description. In Section 5 we recall how, under conditions which turn out to account for the main interesting cases, exact solvability extends to the off-critical regime, and how solutions originally obtained in the scattering framework lead to the computation of universal quantities such as combinations of critical amplitudes. The extension of the formalism to crossover phenomena is illustrated in Section 6. In Section 7 we turn to systems with boundaries, showing how general low-energy properties of two-dimensional field theory yield the exact characterization of phase separation and of the interfacial region, an analysis extended in Section 8 to describe the interaction of an interface with the boundary and the effect of boundary geometry. In Section 9 we show how the particle formalism provides exact asymptotic results for percolation on the rectangle away from criticality, where the methods of boundary conformal field theory do not apply. The last section is devoted to few final remarks.

2. General notions

2.1. Universality

In the framework of classical equilibrium statistical mechanics [1] a system is specified by the Hamiltonian $\mathcal{H}$, whose value is determined by the configuration of the system. The expectation value of an observable $\mathcal{O}$ is the statistical average over configurations

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \sum_{\text{configurations}} \mathcal{O}\, e^{-\mathcal{H}/T}, \tag{1}$$

where $T \geq 0$ is the temperature[1] and the normalization factor (partition function)

$$Z = \sum_{\text{configurations}} e^{-\mathcal{H}/T} \tag{2}$$

ensures that the average of the identity is 1. We are interested in systems which in the limit of infinitely many degrees of freedom undergo a phase transition for some critical value T_c of the temperature. In more than one infinite dimension, this is signaled by an order observable whose expectation value (order parameter) vanishes above T_c and is a function of T below T_c. The transition is said to be of the first order if the order parameter has a discontinuity at T_c, and of the second order (or, more generally, continuous) otherwise.

Phase transitions are normally associated to *spontaneous* symmetry breaking. A group G, mapping configurations into configurations, is a symmetry of the system if $\mathcal{H}$ is left invariant by the action of G. Configurations of minimal energy (ground states) are mapped into each other by G, and below T_c a G-invariant system chooses the phase dominated by a specific ground state as $T \to 0$.

[1] We adopt units in which $k_B = 1$.