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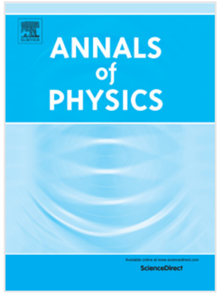

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The Classical and Quantum Mechanics of a Particle on a Knot

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A free particle is constrained to move on a knot obtained by winding around a putative torus. The classical equations of motion for this system are solved in a closed form. The exact energy eigenspectrum, in the thin torus limit, is obtained by mapping the time-independent Schrödinger equation to the Mathieu equation. In the general case, the eigenvalue problem is described by the Hill equation. Finite-thickness corrections are incorporated perturbatively by truncating the Hill equation. Comparisons and contrasts between this problem and the well-studied problem of a particle on a circle (planar rigid rotor) are performed throughout.

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INTRODUCTION

The example of a particle constrained to move along a circle – the so-called planar rigid rotor – is one of the simplest problems that is discussed in text-books of quantum mechanics. The beguiling simplicity of this problem is at the heart of many non-trivial ideas that pervade modern physics. For understanding many issues like the existence of inequivalent quantizations of a given classical system [1], the role of topology in the definition of the vacuum state in gauge theories [2], band structure of solids [3], generalised spin and statistics of the anyonic type [4], and the study of mathematically interesting algebras of quantum observables on spaces with non-trivial topology [5], the problem of a particle on a circle serves as a toy model.

In this paper, we consider the problem of a particle constrained to move on a torus knot. Besides adding a new twist to the aforementioned problems, the present system can be thought of as a double-rotor (analogous to the double-pendulum, but without the gravitational field) which is a genuine non-planar generalization of the planar rotor.

The paper is organised as follows: In the next section we introduce toroidal coordinates in terms of which the constraints which restrict the motion of the particle to the torus knot are most naturally incorporated. As a warm-up, we then analyse the particle on a circle in toroidal coordinates. This prelude allows us to compare and contrast the results of the subsequent sections with the well-known results for the particle on a circle. The following two sections deal with the classical and quantum mechanics of a particle on a torus knot. In the penultimate section we briefly touch upon the possibility of inequivalent quantizations of the particle on a knot. These will be labelled by two parameters, in contrast to the particle on a circle. The concluding section summarises and presents an outlook.

TOROIDAL COORDINATES

The toroidal coordinates [6] are denoted by $0 \le \eta < \infty, \quad -\pi < \theta \le \pi, \quad 0 \le \phi < 2\pi$. Given a toroidal surface of major radius R and minor radius d, we introduce a dimensional parameter a, defined by $a^2 = R^2 - d^2$, and a dimensionless parameter η_0, defined by $\eta_0 = \cosh^{-1}(R/d)$. The equation $\eta =$ constant, say η_0, defines a toroidal surface. The combination R/d is called the aspect ratio. Clearly, larger η_0 corresponds to smaller thickness of the torus. In the limit $\eta_0 \to \infty$, the torus degenerates into a limit circle.

The toroidal coordinates are related to the usual Cartesian coordinates by the equations

$$x = \frac{a\sinh\eta\cos\phi}{(\cosh\eta - \cos\theta)}, \quad y = \frac{a\sinh\eta\sin\phi}{(\cosh\eta - \cos\theta)}, \quad z = \frac{a\sin\theta}{(\cosh\eta - \cos\theta)}. \tag{1}$$

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