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Shape invariant potentials in higher dimensions

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ABSTRACT

In this paper we investigate the shape invariance property of a potential in one dimension. We show that a simple ansatz allows us to reconstruct all the known shape invariant potentials in one dimension. This ansatz can be easily extended to arrive at a large class of new shape invariant potentials in arbitrary dimensions. A reformulation of the shape invariance property and possible generalizations are proposed. These may lead to an important extension of the shape invariance property to Hamiltonians that are related to standard potential problems via space time transformations, which are found useful in path integral formulation of quantum mechanics.

1. Introduction

Supersymmetric quantum mechanics (SUSYQM) [1] played an important role in understanding the problem of solvable potentials. The shape invariance (SI) condition introduced by Gendenshtein [2]

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gives a sufficient condition for the solvability of a given one dimensional potential. This helped in obtaining a whole class of shape invariant potentials (SIPs) which are analytically exactly solvable (ES). It also gave useful insights into the factorization method [3] and has played an important role in the past three decades in constructing new ES potentials in one dimension [4–7]. In addition n parameter dependent SIPs have also been studied in [8]. More recently supersymmetry (SUSY) has provided another route to the construction of rational potentials [9–13], which have the new exceptional orthogonal polynomials (EOPs) as a part of their solutions [14,15].

In higher dimensions the number of known solvable potential models, other than separable ones is severely restricted [16–20]. The power of SUSY and the SI condition give a hope that one may construct a new class of ES models in higher dimensions. At present, work on SI in higher dimensional models is very limited. Use of the SI requirement has been shown to lead to known ES potential models in one dimension [21].

In this paper, we present, an alternate route to analyse and obtain the solutions to SI requirement. This is achieved by the use of a simple ansatz which leads to the solution for the superpotential in terms of the free particle Schrödinger equation solutions. This approach and its extension reproduce all the known SI potentials in one dimension, including those recently discovered potentials related to EOPs [9–15]. In addition, this route to the solutions of the SI requirement trivially generalizes to higher dimensions and leads to, in the first instance, a large class of SI potentials which get related to the solutions of free particle Schrödinger equation in higher dimension.

Even in one dimension, the SI property alone is not sufficient for obtaining exact solutions for the bound state energy eigenvalues and eigenfunctions. In addition one needs to use the intertwining property of partner Hamiltonians. For separable potential models, the intertwining property works exactly in the same way as in one dimension, as the solution of the problem reduces to solution of several one dimensional problems. The known examples of intertwining property in two dimensions indicate that, in practice, this property may have a limited role to play in higher dimensional models and one needs a fresh approach.

A possible route may be the use of space time transformations found useful for obtaining exact solutions of quantum mechanical problems in the path integral approach. In 1984, Duru and Kleinert used space time transformations to provide an exact solution of hydrogen atom problem in three dimensions within the path integral formalism [22,23]. In the next ten years it was shown to be useful for obtaining exact path integral solutions of many other problems [24–30]. Further discussion of space time transformations is given in [26–29]. It is important to mention that the use of space time transformations is not limited to the path integral formalism alone and has been found to be useful beyond one dimensional potential problems. Inspired by the success of the space time transformations, a generalization of shape invariance requirements to Hamiltonians of a more general form is presented. As an example, it is applied to the radial equation for free particle equation in three dimensions to show that this leads to known recurrence relations between spherical Bessel functions.

The paper is organized as follows. In the next section we briefly describe SUSYQM and we give an alternate definition of the SI condition and follow a new approach to obtain solutions of SI requirement. It is shown that all the known ES solvable potentials in one dimension are obtained by making use of few simple ansatz. In Section 3, we show that, in arbitrary dimensions, this route to the analysis of SI condition does not require anything new and easily leads to SIPs. In Section 4, we summarize our results and conclude by giving routes to further generalizations of the SI property.

2. Supersymmetric quantum mechanics

In SUSYQM [1,5], we have a pair of partner potentials $V^{\pm}(x)$ defined in terms of the superpotential $W(x)$ as

$$V^{(\pm)}(x) = W^2(x) \pm W'(x), \tag{1}$$

where the prime denotes differentiation with respect to x and $W(x)$ is defined as

$$W(x) = -\frac{d}{dx} \log \psi_0^{(-)}(x). \tag{2}$$