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Composition of transfer matrices for potentials with overlapping support

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HIGHLIGHTS

- Generalizes the composition rule for transfer matrices to potentials with overlapping support.
- Computes the correction term to the standard composition rule.
- Applies the results in the study of unidirectionally invisible potentials.

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ABSTRACT

For a pair of real or complex scattering potentials $v_j : \mathbb{R} \to \mathbb{C}$ ($j = 1, 2$) with support I_j and transfer matrix $\mathbf{M}_j$, the transfer matrix of $v_1 + v_2$ is given by the product $\mathbf{M}_2\mathbf{M}_1$ provided that I_1 lies to the left of I_2. We explore the prospects of generalizing this composition rule for the cases that I_1 and I_2 have a small intersection. In particular, we show that if I_1 and I_2 intersect in a finite closed interval of length ℓ in which both the potentials are analytic, then the lowest order correction to the above composition rule is proportional to ℓ^5. This correction is of the order of ℓ^3, if v_1 and v_2 are respectively analytic throughout this interval except at $x = \ell$ and $x = 0$. We use these results to explore the superposition of a pair of unidirectionally invisible potentials with overlapping support.

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1. Introduction

Transfer matrices have numerous applications in a variety of scattering problems in physics and engineering [1,2]. This is mainly because of their composition property that allows for the determination of the scattering properties of a complicated system from the contributions of its simpler constituents. This is most simply described in the standard one-dimensional potential scattering [3].

Consider a possibly complex-valued scattering potential $v : \mathbb{R} \to \mathbb{C}$ with an asymptotic decay rate such that [4]

$$\int_{-\infty}^{\infty} (1 + |x|)|v(x)|dx < \infty, \tag{1}$$

and let k be a positive real (wave)number. Then every solution of the Schrödinger equation

$$-\psi''(x) + v(x)\psi(x) = k^2\psi(x), \quad x \in \mathbb{R}, \tag{2}$$

satisfies

$$\psi(x) \to A_\pm e^{ikx} + B_\pm e^{-ikx} \quad \text{as } x \to \pm\infty, \tag{3}$$

where $A_\pm$ and $B_\pm$ are possibly k-dependent complex coefficients [3]. The transfer matrix $\mathbf{M}$ of the potential v is a k-dependent 2×2 matrix that fulfils the relation [1]

$$\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = \mathbf{M} \begin{bmatrix} A_- \\ B_- \end{bmatrix}. \tag{4}$$

Similarly to the S-matrix, $\mathbf{M}$ encodes the scattering properties of the potential v. Recalling that scattering solutions of (2) are given in terms of the left/right reflection and transmission amplitudes, $R^{l/r}$ and T, according to

$$\psi_k^l(x) = \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{for } x \to -\infty, \\ Te^{ikx} & \text{for } x \to \infty, \end{cases} \qquad \psi_k^r(x) = \begin{cases} Te^{-ikx} & \text{for } x \to -\infty, \\ e^{-ikx} + R^r e^{ikx} & \text{for } x \to \infty, \end{cases} \tag{5}$$

and using (3)–(5), we can express the entries M_{ij} of $\mathbf{M}$ as [5]

$$\begin{aligned} M_{11} &= T - R^l R^r/T, & M_{12} &= R^r/T, \\ M_{21} &= -R^l/T, & M_{22} &= 1/T. \end{aligned} \tag{6}$$

These, in particular, imply that $\det \mathbf{M} = 1$. We also note that (1) is a sufficient condition for the (global) existence of the Jost solutions, $\psi_{k+} = \psi_k^l/T$ and $\psi_{k-} = \psi_k^r/T$, of (2), [6].

Now, suppose that v can be written as the sum of a pair of scattering potentials $v_j : \mathbb{R} \to \mathbb{C}$ ($j = 1, 2$) with the same asymptotic decay property as v, such that the support[1] of v_1 lies to the left of that of v_2. Using I_j to label the support of v_j, we express this condition by '$I_1 \prec I_2$'.[2] Under this assumption, we can relate $\mathbf{M}$ to the transfer matrix $\mathbf{M}_j$ of v_j according to

$$\mathbf{M} = \mathbf{M}_2\mathbf{M}_1. \tag{7}$$

This is the celebrated 'composition property' of the transfer matrix. It is also called the 'group property', because transfer matrices belong to the matrix group $SL(2, \mathbb{C})$ and (7) involves the group multiplication for this group [2]. The primary aim of this article is to seek for a generalization of (7) to the cases where $I_1 \cap I_2$ is a finite interval. Without loss of generality, we can identify the latter with $[0, \ell]$, where ℓ is a real parameter signifying the length of $I_1 \cap I_2$, and demand that for all $x_1 \in I_1 \setminus I_2$ and $x_2 \in I_2$, $x_1 < x_2$. We denote this relation by '$I_1 \preccurlyeq I_2$'. Fig. 1 provides a schematic description of this condition. To summarize, we wish to generalize the composition property (7) of the transfer matrix, which holds for $I_1 \prec I_2$, to situations where $I_1 \preccurlyeq I_2$.

1 The support of a function $f : \mathbb{R} \to \mathbb{C}$ is the smallest closed interval outside which f vanishes.

2 '$I_1 \prec I_2$' means that for every $x_1 \in I_1$ and $x_2 \in I_2$, we have $x_1 \le x_2$.