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A nodal domain theorem for integrable billiards in two dimensions

Rhine Samajdar [a], Sudhir R. Jain [b,*]

[a] *Indian Institute of Science, Bangalore 560012, India*

[b] *Nuclear Physics Division, Bhabha Atomic Research Centre, Mumbai 400085, India*

HIGHLIGHTS

- We find that the number of nodal domains of eigenfunctions of integrable, planar billiards satisfy a class of difference equations.
- The eigenfunctions labelled by quantum numbers (m, n) can be classified in terms of m mod kn.
- A theorem is presented, realising algebraic representations of geometrical patterns exhibited by the domains.
- This work presents a connection between integrable systems and difference equations.

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ABSTRACT

Eigenfunctions of integrable planar billiards are studied — in particular, the number of nodal domains, ν of the eigenfunctions with Dirichlet boundary conditions are considered. The billiards for which the time-independent Schrödinger equation (Helmholtz equation) is separable admit trivial expressions for the number of domains. Here, we discover that for all separable and non-separable integrable billiards, ν satisfies certain difference equations. This has been possible because the eigenfunctions can be classified in families labelled by the same value of m mod kn, given a particular k, for a set of quantum numbers, m, n. Further, we observe that the patterns in a family are similar and the algebraic representation of the geometrical nodal patterns is found. Instances of this representation are explained in detail to understand the beauty of the patterns. This paper therefore presents a mathematical connection between integrable systems and difference equations.

* Corresponding author. Tel.: +91 22 25593589.
E-mail address: srjain@barc.gov.in (S.R. Jain).

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1. Introduction

The 'particle in a box' has served as a model for understanding various phenomena in solid state and nuclear physics — the theory of dynamical systems terms these as 'billiards'. Studies of their energy spectra and eigenfunctions, and their connections with quantum chaos have been very fruitful and exciting. One of the properties of the eigenfunctions of these billiards is the organisation of regions with positive and negative signs. These domains appear in rather complex forms [1]; their number displays a near-incomprehensible order if organised in increasing energy. We are familiar with domains that appear in a system which has two states or phases. For instance, in magnetic materials, there are regions of positively and negatively aligned spins. Their shapes and areas promise interesting statistical questions. There has been a lot of interest in studying the nodal domain statistics in recent times of billiards in two dimensions [2]. Here, we present a general result for the number of nodal domains of integrable plane polygonal billiards — the geometries are rectangle, circle, ellipse, and triangles with angles $(45, 45, 90)$, $(30, 60, 90)$ and $(60, 60, 60)$. The great interest in these systems emerges from the simplicity they seem to present, and their ubiquity in a large number of contexts.

The Schrödinger equation for a particle inside a rectangular box satisfying Dirichlet or Neumann boundary conditions can be immediately solved. The eigenfunctions are the well-known product of two sine or cosine functions and the nodal domains make a checkerboard. For the right isosceles and the equilateral triangle billiards, which are non-separable, the solutions are respectively two and three terms, each a product of two sinusoidal functions. For the equilateral triangle, we discovered a difference equation satisfied by the number of nodal domains, $\nu_{m,n}$ [3], a discrete variable of the system. In this article, we show that $\nu_{m,n}$ for all the integrable billiards obey similar difference equations. This general result is amazing, in view of widely varied findings observed for these billiards about various statistical measures [4–6]. Interestingly, recent investigations of integrable lattice systems [7] have also sought to probe the intricate connection between the theory of exactly integrable discrete systems and the formalism of difference equations.

2. General mathematical formulation

Let $\mathcal{D} \subset \mathbb{R}^2$ be a compact, connected domain on a surface with a smooth Riemannian metric in two dimensions. Assuming Dirichlet conditions along the boundary $\partial\mathcal{D}$ and denoting the Laplace–Beltrami operator by ∇^2, the eigenvalue problem is formulated as

$$-\nabla^2\psi_j = E_j\psi_j \quad \text{and} \quad \psi_j|_{\partial\mathcal{D}} = 0. \tag{2.1}$$

A nodal domain of the wavefunction ψ_j is a connected domain in $\mathcal{D}$ where $\psi_j \neq 0$, which therefore defines a maximally connected region wherein the function does not change sign. The subscript j simply denotes the ordering of the spectrum such that $E_j \leq E_{j+1}$. The importance of the nodal set arises from the fact that the sequence of the number of nodal domains of the eigenfunctions of the Schrödinger equation not only bears significant geometric information about the system [8] but also provides a new criterion for chaos in quantum mechanics [4]. Hence, it may be reasonable to conjecture that the difference equations also encode the geometry of the system itself.

Any wavefunction ψ_j on the domain $\mathcal{D}$ is characterised by two quantum numbers for manifolds on $\mathbb{R}^2$ which are represented hereafter as m and n, unless specified otherwise, where $m, n \in \mathbb{N}$. Let $\nu_{m,n}$ denote the total number of nodal domains of the wavefunction $\psi_{m,n}$. Furthermore, let $R_{k,n}$ be an equivalence relation defined on the set of wavefunctions as

$$R_{k,n} = \{(\psi(m_1, n), \psi(m_2, n)) : m_1 \equiv m_2 (\mathrm{mod}\ kn)\}. \tag{2.2}$$

The relation $R_{k,n}$ defines a partition $\mathcal{P}$ of the set of wavefunctions into equivalence classes $[\mathcal{C}_{kn}]$ where $\mathcal{C}_{kn} = m \mod kn$. Here, the parameter k represents the number of linearly independent terms of which the wavefunction $\psi_{m,n}$ is a sum. Consideration of the sequence of $\nu_{m,n}$ for wavefunctions that belong to the same class illustrates the rich structure of the difference equations that arise in two-dimensional integrable billiards. To this end, we introduce the forward difference operator, $\Delta_t\mathcal{F}(x_1, x_2)$, operating on the first index x_1 of a generic function $\mathcal{F}$, with a finite difference t, i.e. $\Delta_t\mathcal{F}(x_1, x_2) = \mathcal{F}(x_1 + t, x_2) - \mathcal{F}(x_1, x_2)$.