Annals of Physics 351 (2014) 138–151

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Non Abelian structures and the geometric phase of entangled qudits

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ARTICLE INFO

Article history:
Received 10 June 2014
Accepted 27 August 2014
Available online 6 September 2014

Keywords:
Geometric phases
Entanglement
Lie algebras

ABSTRACT

In this work, we address some important topological and algebraic aspects of two-qudit states evolving under local unitary operations. The projective invariant subspaces and evolutions are connected with the common elements characterizing the $\mathfrak{su}(d)$ Lie algebra and their representations. In particular, the roots and weights turn out to be natural quantities to parametrize cyclic evolutions and fractional phases. This framework is then used to recast the coset contribution to the geometric phase in a form that generalizes the usual monopole-like formula for a single qubit.

1. Introduction

Entanglement is an essential component in quantum information protocols. The ability to operate entangled states without destroying their main features is often the central task in experimental implementations. Pure state entanglement can be measured by the concurrence [1], which is insensitive to local unitary operations on the individual subsystems. Under these evolutions, the geometric phase acquired by maximally entangled pairs of qubits has been predicted to occur in discrete steps [2–4]. This discussion has been recently extended to multiple qubits [5]. Phase steps, where a factor $e^{i\pi}$ is introduced, have been experimentally demonstrated with qubits encoded on spin–orbit laser modes [6], nuclear spins [7] and entangled photon pairs [8].

In Ref. [9], based on the kinematic approach developed by Mukunda and Simon [10,11], we investigated entangled qudits under unitary local operations, identifying some geometrical and topological

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http://dx.doi.org/10.1016/j.aop.2014.08.021

aspects. In particular, the geometric phase was calculated in terms of the concurrence, and fractional phases in cyclic evolutions were identified and analyzed. The extension to pairs of qudits with different dimensions was done in Ref. [12], where the overlap of the evolving and initial state was illustrated with numerical examples in two-qutrit and qubit–qutrit systems.

Experimental setups for the observation of fractional phases for entangled qudits [13] and multiple qubits [14] have already been proposed. Quantum gates based on geometric phases have been studied in the literature as a robust means for quantum computation [15,16]. In addition, fractional phases have been conjectured as a possible resource for fault tolerant quantum computation, though associated with a different physical situation. Namely, the fractional statistics due to the multiply connected nature of the configuration space of anyons [17]. Because of the various experimental and theoretical contexts involved, it is worth seeking for a thorough understanding of entangled qudit pairs operated by local unitary evolutions.

In this work, we provide further insight into the different mathematical aspects involved. Initially, we shall obtain the fundamental homotopy group for the projective space of separable states and that for general rank-d states. Next, we will show how the $\mathfrak{su}(d)$ Lie algebra structure provides the appropriate tools to characterize two-qudit states, fractional and geometric phases. For example, evolutions containing a Cartan factor along the weights of $SU(d)$ representations are those generating fractional phases. Moreover, by using a time-dependent Lie algebra basis, we will show how to write the coset contribution to the geometric phase in terms of local Cartan elements n_q, $q = 1, \ldots, d-1$, projected along the fundamental weights of $\mathfrak{su}(d)$. These contributions will correspond to a generalization of the monopole-like Berry phase for a single qubit, where the phase can be expressed as the flux of a topological charge density on S^2, for an $S^2 \to \hat{n} \in S^2$ mapping. The mathematics involved turns out to be that needed to discuss center vortices [18–22] and non Abelian monopoles [23] in Yang–Mills–Higgs models with $SU(d) \to Z(d)$ spontaneous symmetry breaking.

In Section 2, we review some general properties of invariant projective subspaces and compute the fundamental homotopy groups for separable and rank-d pure states. In Section 3, we relate local evolutions and non Abelian connections, defined in terms of a local Lie algebra basis. In that section, we also show what are the possible evolutions leading to fractional phases. Section 4 is devoted to identify the Mukunda–Simon geometric phase as a sum weighted by d invariants under local evolutions, as well as by the weights of the fundamental $SU(d)$ representation. This phase is then carefully worked out to recast the coset sector as a superposition of monopole-like contributions. Finally, in Section 5 we present our conclusions.

2. The topology of invariant subspaces

In Quantum Mechanics, an important concept is that of the projective space of states, which is essentially a topological space such that different points represent physically distinct quantum states. This space can be obtained by considering the equivalence relation between (normalized) kets,

$$|\psi\rangle \sim |\psi'\rangle \quad \text{if } |\psi\rangle' = e^{if}|\psi\rangle, \tag{1}$$

which induces a partition of the Hilbert space into equivalence classes, and then identifying points within each class to form a quotient space. For a Hilbert space $\mathcal{H}_n$ of complex dimension n, the associated projective space is the manifold CP^{n-1}, whose real dimension is $2n-1$. As the group of unitary transformations $U(n)$ acts transitively on $\mathcal{H}_n$, CP^{n-1} can also be written as the quotient of $U(n)$ by the stability group associated with any state vector $|\psi_0\rangle$. Alternatively, noting that $\mathbb{U} = e^{i\phi}\,\bar{\mathbb{U}}$, with $\bar{\mathbb{U}} \in SU(n)$, CP^{n-1} is the quotient of $SU(n)$ by the stability group $H \subset SU(n)$ that leaves the ket $|\psi_0\rangle$ invariant (up to a phase),

$$H = \{h \in SU(n)/h|\psi_0\rangle = e^{i\chi}|\psi_0\rangle\}. \tag{2}$$

As is well-known, H is isomorphous to $U(n-1)$,

$$h = \begin{pmatrix} (\det u)^{-1} & \mathbb{O}_{1\times(n-1)} \\ \mathbb{O}_{(n-1)\times 1} & u \end{pmatrix}, \quad u \in U(n-1), \tag{3}$$