Annals of Physics 351 (2014) 382–406

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

A first class constraint generates not a gauge transformation, but a bad physical change: The case of electromagnetism

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HIGHLIGHTS

- A first-class constraint changes the electric field E, spoiling Gauss's law.
- A first-class constraint does not leave the action invariant or preserve q, $0 - dH/dp$.
- The gauge generator preserves E, q, $0 - dH/dp$, and the canonical action.
- The error in proofs that first-class primaries generating gauge is shown.
- Dirac's conjecture about secondary first-class constraints is blocked.

ARTICLE INFO

Article history:
Received 14 March 2014
Accepted 20 August 2014
Available online 6 September 2014

Keywords:
Dirac–Bergmann constrained dynamics
Gauge transformations
Canonical quantization
Observables
Hamiltonian methods
First-class constraints

ABSTRACT

In Dirac–Bergmann constrained dynamics, a first-class constraint typically does not *alone* generate a gauge transformation. By direct calculation it is found that each first-class constraint in Maxwell's theory generates a change in the electric field $\vec{E}$ by an arbitrary gradient, spoiling Gauss's law. The secondary first-class constraint $p^i{}_{,i} = 0$ still holds, but being a function of derivatives of momenta (mere auxiliary fields), it is not directly about the observable electric field (a function of derivatives of A_μ), which couples to charge. Only a special combination of the two first-class constraints, the Anderson–Bergmann–Castellani gauge generator G, leaves $\vec{E}$ unchanged. Likewise only that combination leaves the canonical action invariant—an argument independent of observables. If one uses a first-class constraint to generate instead a canonical transformation, one partly strips the canonical coordinates of physical meaning as electromagnetic potentials, vindicating the Anderson–Bergmann Lagrangian orientation of interesting canonical transformations. The need to keep gauge-invariant the relation $\dot{q} - \frac{\delta H}{\delta p} = -E_i - p^i = 0$ supports using the gauge

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http://dx.doi.org/10.1016/j.aop.2014.08.014

generator and primary Hamiltonian rather than the separate first-class constraints and the extended Hamiltonian.

Partly paralleling Pons's criticism, it is shown that Dirac's proof that a first-class primary constraint generates a gauge transformation, by comparing evolutions from *identical* initial data, cancels out and hence fails to detect the alterations made to the initial state. It also neglects the arbitrary coordinates multiplying the secondary constraints *inside* the canonical Hamiltonian. Thus the gauge-generating property has been ascribed to the primaries alone, not the primary–secondary team G. Hence the Dirac conjecture about secondary first-class constraints as generating gauge transformations rests upon a false presupposition about primary first-class constraints. Clarity about Hamiltonian electromagnetism will be useful for an analogous treatment of GR.

1. Introduction

In the early stages of research into constrained Hamiltonian dynamics by Bergmann's school and earlier by Rosenfeld [1], it was important to ensure that the new Hamiltonian formalism agreed with the established Lagrangian formalism. That was very reasonable, for what other criteria for success were there at that stage? One specific manifestation of Hamiltonian–Lagrangian equivalence was the recovery of the usual 4-dimensional Lagrangian gauge transformations for Maxwell's electromagnetism and (more laboriously) GR by Anderson and Bergmann [2]. 4-dimensional Lagrangian-equivalent gauge transformations were implemented by Anderson and Bergmann in the Hamiltonian formalism using the gauge generator (which I will call G), a specially tuned sum of the first-class constraints, primary and secondary, in electromagnetism or GR [2].

At some point, equivalence with 4-dimensional Lagrangian considerations came to play a less significant role. Instead the idea that a first-class constraint *by itself* generates a gauge transformation became increasingly prominent. That claim [3] has been called the "'standard'" interpretation [4] and is officially adopted in Henneaux and Teitelboim's book [5, pp. 18–54] (at least nominally, though not always in reality [6]) and in countless other places [7–9]. This idea displaced the Anderson–Bergmann gauge generator until the 1980s and remains a widely held view, though no longer a completely dominant one in the wake of the Lagrangian-oriented reforms of Castellani, Sugano, Pons, Salisbury, Shepley, *etc.* Closely paralleling the debate between the Lagrangian-equivalent gauge generator G and the distinctively Hamiltonian idea that a first-class constraint generates a gauge transformation is the debate between the Lagrangian-equivalent primary Hamiltonian H_p (which adds to the canonical Hamiltonian H_c all the primary constraints, whether first- or second-class) and Dirac's extended Hamiltonian H_E, which adds to the primary Hamiltonian the first-class secondary constraints.

A guiding theme of Pons, Shepley, and Salisbury's series of works [10–12] is important:

> We have been guided by the principle that the Lagrangian and Hamiltonian formalisms should be equivalent …in coming to the conclusion that they in fact are. [13]

While proponents of the primary Hamiltonian H_p have emphasized the value of making the Hamiltonian formalism equivalent to the Lagrangian, what has perhaps been lacking until now is an effective argument that the Lagrangian-inequivalent extended Hamiltonian is erroneous. While inequivalence of the extended Hamiltonian to the Lagrangian might seem worrisome, it is widely held that the difference is confined to gauge-dependent unobservable quantities and hence makes no real physical difference. If that claim of empirical equivalence were true, it would be a good defense of the permissibility of extending the Hamiltonian. But is that claim of empirical equivalence true?