Annals of Physics 351 (2014) 935–951

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

On systems having Poincaré and Galileo symmetry

Peter Holland

Green Templeton College, University of Oxford, Oxford OX2 6HG, England, United Kingdom

ARTICLE INFO

Article history:
Received 20 September 2014
Accepted 11 October 2014
Available online 19 October 2014

Keywords:
Chaplygin
Quantum hydrodynamics
Carroll group
Field-dependent symmetry
Noether
Electromagnetism

ABSTRACT

Using the wave equation in $d \geq 1$ space dimensions it is illustrated how dynamical equations may be simultaneously Poincaré and Galileo covariant with respect to different sets of independent variables. This provides a method to obtain dynamics-dependent representations of the kinematical symmetries. When the field is a displacement function both symmetries have a physical interpretation. For $d = 1$ the Lorentz structure is utilized to reveal hitherto unnoticed features of the non-relativistic Chaplygin gas including a relativistic structure with a limiting case that exhibits the Carroll group, and field-dependent symmetries and associated Noether charges. The Lorentz transformations of the potentials naturally associated with the Chaplygin system are given. These results prompt the search for further symmetries and it is shown that the Chaplygin equations support a nonlinear superposition principle. A known spacetime mixing symmetry is shown to decompose into label-time and superposition symmetries. It is shown that a quantum mechanical system in a stationary state behaves as a Chaplygin gas. The extension to $d > 1$ is used to illustrate how the physical significance of the dual symmetries is contingent on the context by showing that Maxwell's equations exhibit an exact Galileo covariant formulation where Lorentz and gauge transformations are represented by field-dependent symmetries. A natural conceptual and formal framework is provided by the Lagrangian and Eulerian pictures of continuum mechanics.

E-mail address: peter.holland@gtc.ox.ac.uk.

http://dx.doi.org/10.1016/j.aop.2014.10.009

1. Introduction

That a system of equations could admit both Lorentz and Galileo boosts as exact symmetries seems nonsensical at first sight. The relation between the two is customarily presented as one of asymptotic inclusion, the Galilean transformation emerging from the Lorentzian in the limit of low velocities relative to the speed of light. Nevertheless, it is well known that a dynamical equation can be both Poincaré and Galileo covariant if, for example, it possesses the general linear group as a symmetry (of which the Poincaré and Galileo groups are subgroups; for an example see [1]). Here we examine how a physical system may be simultaneously 'relativistic' and 'non-relativistic' by virtue of admitting descriptions in terms of two sets of independent variables, its governing equations being Poincaré covariant with respect to one set and Galileo covariant with respect to the other. Since the (Lorentz or Galileo) boost symmetry pertaining to one set of variables has a realization in the other set, each set of variables exhibits two independent symmetries involving velocity parameters. This approach then provides a method to discover alternative representations of the kinematical symmetries, which may become dynamical by virtue of depending on solutions of the dynamical equations. The extent to which the two symmetries describe operations involving physical observers, or have a more arcane character, depends on the meaning afforded to the variables chosen to describe the system that are subject to the relevant transformations and the nature of the governing equations. Obviously, for a given set of equations at most one of the velocity parameters may characterize a spatial boost, i.e., a transformation between inertial reference frames in uniform relative motion. The other parameter must then relate to a different notion of 'boost'.

Although it does not seem to be widely known, this dual covariance is a feature of field equations used in physics and is exemplified by the wave equation in $d \geq 1$ space dimensions, which will be our focus. When the field variable is a displacement function this provides an example where both symmetries have a physical interpretation. We shall examine the $d = 1$ case in detail as this already illustrates the key symmetry properties, which persist in higher dimensions but take more complex forms. The cases we consider are contained in a class of theories where each set of independent variables spans a $(d, 1)$-dimensional coordinate-time space, with a common time coordinate. A natural conceptual and formal framework is provided by continuum mechanics, in which context the sets of envisaged variables correspond to the Lagrangian and Eulerian pictures. The variables are linked by the standard transformation rules connecting the two descriptions, where fields in one picture become independent variables in the other.

In elementary treatments of the derivation of continuous field equations from discrete mechanical models, the wave equation is obtained as the limit of the Newtonian dynamics of a linear chain of coupled harmonic oscillators [2]. Denoting the limiting particle label by $a \in \mathbb{R}$ (having the dimension of length) and assuming the limiting mass density k of the oscillators in the reference state is uniform, one obtains for the longitudinal displacement $q(a, t)$ of the ath particle in a one-dimensional chain (assumed infinite) at time t:

$$\frac{\partial^2 q}{\partial t^2} - c^2 \frac{\partial^2 q}{\partial a^2} = 0 \tag{1.1}$$

where $c = \sqrt{Y/k}$ and Y is Young's modulus. Note that this equation holds for finite amplitude q. The description is a *Lagrangian*, or *material*, one [3], which specifies the state of a system by the motion of each of the mass points (independent variables a, t; state variable position $q(a, t)$). The description is completed by the law of mass conservation whose solution gives the density at time t: $\rho(q(a, t), t) = k(\partial q/\partial a)^{-1}$. Following the usual approach of continuum mechanics, we may consider an alternative description, the *Eulerian*, or *spatial*, one, which specifies the state at a fixed spacetime point (independent variables $x = q(a, t), t$; state variables density $\rho(x, t)$ and velocity $v(x, t)$). As we shall see, the Lagrangian and Eulerian formulations each exhibit relativistic and non-relativistic symmetries. For this system, the covariance of the field equations in the two pictures with respect to a physical boost is described by Galileo's transformation, not Lorentz's. The latter describes a time-dependent label-time substitution.

It was observed by Earnshaw [4] in 1860 that the Eulerian equation of state corresponding to the process (1.1) is $p(\rho) = A - 2\lambda/\rho$, $\lambda = \frac{1}{2}c^2k^2 > 0$ (for a review see [5]). Several authors [4,6,7] have