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Scattering amplitudes for multi-indexed extensions of solvable potentials

C.-L. Ho [a], J.-C. Lee [b,*], R. Sasaki [c,d]

[a] *Department of Physics, Tamkang University, Tamsui 251, Taiwan, ROC*
[b] *Department of Electrophysics, National Chiao-Tung University, Hsinchu, Taiwan, ROC*
[c] *Center for Theoretical Sciences, National Taiwan University, Taipei, Taiwan, ROC*
[d] *Department of Physics, Shinshu University, Matsumoto 390-8621, Japan*

HIGHLIGHTS

- Scattering amplitudes calculated for infinitely many new solvable potentials.
- New scattering potentials obtained by deforming six known solvable potentials.
- Multiple Darboux transformations in terms of (pseudo) virtual states employed.
- Scattering amplitudes checked to obey the shape invariance relation.
- Errors in scattering amplitudes of some undeformed potentials in the literature corrected.

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ABSTRACT

New solvable one-dimensional quantum mechanical scattering problems are presented. They are obtained from known solvable potentials by multiple Darboux transformations in terms of virtual and pseudo virtual wavefunctions. The same method applied to confining potentials, *e.g.* Pöschl–Teller and the radial oscillator potentials, has generated the *multi-indexed Jacobi and Laguerre polynomials*. Simple multi-indexed formulas are derived for the transmission and reflection amplitudes of several solvable potentials.

1. Introduction

We address the problem of rational extensions of solvable one-dimensional quantum mechanical scattering problems. Study of exactly solvable potentials in one dimensional quantum mechanics

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[*] Corresponding author.
E-mail address: jcclee@cc.nctu.edu.tw (J.-C. Lee).

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[1–3] has been rapidly developing in recent years. It culminated in the discovery of multi-indexed Jacobi and Laguerre polynomials [4,5] together with the exceptional orthogonal polynomials [6–17] as the main part of the eigenfunctions for the rationally extended Pöschl–Teller and radial oscillator potentials. The main focus of these papers [4–17] has been iso-spectral deformations of the (semi) confining potentials and the bound state eigenfunctions. These rational extensions (deformations) are achieved by multiple Darboux transformations [18–20] in terms of *polynomial type seed solutions*, which are called the *virtual and pseudo virtual state wavefunctions* [4,21,22]. These seed solutions are obtained from the eigenfunctions by discrete symmetry operations (twisting) [4,22] or adopting the same function forms of the eigenfunctions with their degrees much higher than the highest eigenlevel $n_{\max}$ (*overshoot eigenfunctions*) [21]. A Darboux transformation in terms of a virtual state wavefunction generates an iso-spectral deformation, whereas a pseudo virtual state wavefunction used in a Darboux transformation will create an eigenstate at its energy, which is below the original groundstate. Therefore the latter generates a non-isospectral deformation. The multi-index consists of the degrees of the used polynomial type seed solutions. In this paper we simply use the same method (multiple Darboux transformations) and the same polynomial type seed solutions to enlarge the list of exactly solvable scattering problems, starting from five potentials having a finite number of eigenlevels, *e.g.* the Rosen–Morse potential, etc. [21] and the Coulomb potential with the centrifugal barrier. These six potentials are divided into three large groups, Group (A) for which the scattering waves extend from $x = -\infty$ to $x = +\infty$ and Group (B) for which the waves reach $x = +\infty$ but they cannot reach $x = -\infty$, and Group (C) for the long-ranged Coulomb potential on half-line. The solvable full line scattering and half line scattering have very different characteristics. In Group (A), the full line scattering cases, the transmission and reflection amplitudes are invariant under the discrete symmetry transformation, whereas in Groups (B) and (C), the discrete symmetry of the potential does not imply the invariance of the reflection amplitude. The extended scattering data (the transmission and reflection amplitudes) are determined solely by the *asymptotic exponents* of the seed solutions, whereas the deformed potentials, in particular their regularity or non-singularity, depend heavily on the local behaviours of the seed solutions.

Historically, extensions of solvable scattering problems had been discussed in connection with the soliton theory and the inverse scattering problem [23]. The adopted methods had been related to Abraham–Moses transformations [24,25] and the extensions were non-polynomial or non-rational [26–31]. Compared to those non-rational extensions, the multi-indexed extensions reported in this paper are much clearer in notion and simpler in execution, although their applicability is limited to shape invariant potentials only. Two papers discussing similar targets but with limited contents appeared recently [32].

The present paper is organised as follows. In section two the basic structures of the scattering problems and their extensions by multiple Darboux transformations are recapitulated. The full line scattering, called Group (A) and the half line scattering, Groups (B) and (C), are treated separately. Various properties of the polynomial type seed solutions, the virtual state, pseudo virtual state wavefunctions and the overshoot eigenfunctions are explored in section three. The conditions for the non-singularity of the deformed potentials are explained. Sections four, five and six provide the explicit data of the original solvable potentials and the characteristics of the deformed ones. The three potentials belonging to Group (A) are presented in section four: Rosen–Morse Section 4.1, soliton Section 4.2 and hyperbolic symmetric top potential Section 4.3. Those belonging to Group (B) are explained in section five: Morse Section 5.1, Eckart Section 5.2, hyperbolic Pöschl–Teller Section 5.3. The long range Coulomb potential is the sole member of Group (C) summarised in Section 6. Section seven is for a summary and comments.

2. Scattering problems and their extensions

Here we review the setting of one-dimensional quantum mechanical scattering problems and their extensions by multiple Darboux transformations in terms of polynomial type seed solutions. Let a quantum mechanical Hamiltonian $\mathcal{H}$ be defined in an interval $x_1 < x < x_2$, with a smooth potential:

$$\mathcal{H} = -\frac{d^2}{dx^2} + U(x), \quad U(x) \in \mathbb{R}. \tag{2.1}$$