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Dynamics of zero-energy nonspreading non-Gaussian wave packets for a class of central potentials

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HIGHLIGHTS

- Central potentials are considered.
- Nonspreading, non-Gaussian wave packets are constructed.
- Time evolution of the zero-energy packets is studied.
- Quantum–classical correspondence is discussed.

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ABSTRACT

Zero-energy wave packets, coherent states, are constructed in such a way that they retain their shape during the time evolution for a large class of central potentials. The packets are not of the Gaussian type with $-r^2$ dependence but, instead, their shape is determined by $-r^{\frac{1}{\mu+1/2}}$ with $-1/2 < \mu < 1/2$. A very close quantum–classical correspondence is also shown, i.e., the well localized states travel along suitable classical trajectories.

1. Introduction

The most classical among pure states of quantum mechanics, called after Glauber [1] *the coherent states*, are used in many areas of physics from solid state physics through quantum optics to modern cosmology. The problem of construction of such well-localized, nonspreading states was first addressed in 1926 by Schrödinger who successfully solved it for the harmonic oscillator [2], however,

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without a similar result for the hydrogen atom. Since then, numerous methods were developed to derive localized states, wave packets, which travel along classical orbits for variety of potentials [3–7].

Three groups of methods are among the most popular ones. In Barut's approach [3] coherent states are constructed as eigenstates of the lowering operator. They can also be found by acting on a reference state, often chosen as the ground state, with a suitably defined displacement operator. These are Perelomov type coherent states [4]. In the third approach, let us call it Nieto's method [5], the states are constructed as minimum-uncertainty wave packets. Only for the harmonic oscillator the three methods lead to the same wave packet. Otherwise, the methods are not equivalent. A comprehensive review of other approaches can be found in [6,7].

Experimentally, the packets were first created some 25 years ago [8,9] by photoexcitation using ultrashort pulses. The technique was further developed by Jones et al. [10] by using half-cycle pulses. A procedure to produce well-localized Bohr-like wave packets to mimic an electron moving in a Kepler orbit was suggested in [11] and successfully applied in [12]. Some other interesting experimental papers on Rydberg wave packets in atoms can be found in [13] and those for molecules in [14].

In this paper, we construct spatially well-localized and nonspreading packets using the method described in [15]. In the method, one uses superpositions of states with well-defined angular momentum corresponding to the total energy $E = 0$. Calculations are performed in the two-dimensional (2D) configural space for the potentials in the form

$$V_\mu(r) = \left(\frac{\lambda}{2\mu+1}\right)^2 \left[\frac{\lambda^2/2}{(r/r_0)^{\frac{2\mu-1}{\mu+1/2}}} - \frac{Q}{(r/r_0)^{\frac{2\mu}{\mu+1/2}}}\right], \tag{1}$$

where the parameters λ, r_0, Q are positively determined and $\mu \neq -1/2$. It was shown in [16] that these potentials have normalizable $E = 0$ bound states, and that the corresponding wave functions can be deduced from $E \neq 0$ solutions of the 3D harmonic oscillator. This was done with the help of some point transformations.

Our aim is to derive wave packets that move precisely along classical trajectories for the potentials in Eq. (1) for some distinguished values of μ. To this end, in Section 2, we will first find and discuss classical trajectories of the problem. Since for the central potentials the angular momentum is a conserved quantity, the orbits are flat, and therefore we can consider only the 2D case. Then, in Section 3, we shall derive square-integrable wave functions for the radial 2D Schrödinger equation with the potential $V_\mu(r)$. In Section 4, the functions are used for the construction of a stationary non-Gaussian wave packet. Its motion along classical solutions is discussed in Section 5, and finally, our conclusions are given in Section 6.

2. Classical solutions

The $E = 0$ orbits in the potential in Eq. (1) can be easily found using the polar coordinates r and φ. For the angular momentum L and the threshold energy, we now have

$$L = mr^2\dot{\varphi}, \tag{2}$$

$$0 = (m/2)(\dot{r}^2 + r^2\dot{\varphi}^2) + V_\mu(r), \tag{3}$$

where the dots over symbols mean time derivatives. From these equations, we get

$$\frac{dr}{d\varphi} = \sqrt{-r^2 - \frac{2mr^4}{L^2}V_\mu(r)}. \tag{4}$$

Eq. (4) can be integrated if the transformation $\rho = r/r_0 = Z^{\mu+1/2}$ is made. In this way, we obtain

$$\rho_\mu(\varphi) = \left[\frac{2}{B - (B^2 - 4A)^{1/2}\sin\left(\frac{\varphi-\varphi_0}{\mu+1/2}\right)}\right]^{\mu+1/2}, \tag{5}$$