Annals of Physics 337 (2013) 34–48

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Generalized Weyl quantization on the cylinder and the quantum phase

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HIGHLIGHTS

- The generalized Weyl quantization on the cylindrical phase space is formulated.
- A self-adjoint phase operator on the Hilbert space of the square integrable functions on the circle is given.
- A new uncertainty relation between the quantum phase and the number operator is found.

ARTICLE INFO

Article history:
Received 21 February 2013
Accepted 20 June 2013
Available online 28 June 2013

Keywords:
Stratonovich–Weyl quantization on the cylinder
Quantum phase
The Pegg–Barnett approach

ABSTRACT

Generalized Weyl quantization formalism for the cylindrical phase space $S^1 \times \mathbb{R}^1$ is developed. It is shown that the quantum observables relevant to the phase of the linear harmonic oscillator or electromagnetic field can be represented within this formalism by the self-adjoint operators on the Hilbert space $L^2(S^1)$.

1. Introduction

The problem of defining the phase operator for a harmonic oscillator or for a single-mode electromagnetic field in quantum mechanics is an intriguing and still unsolved question. The existence of a self-adjoint phase operator $\hat{\Phi}$ canonically conjugate to the number operator $\hat{N}$

$$\left[\hat{\Phi}, \hat{N}\right] = -i \tag{1}$$

was postulated by Dirac about 85 years ago [1]. However, in 1964 Susskind and Glogower [2] showed that Dirac's assumption led to essential controversies (see also [3,4]). In conclusion, instead of $\hat{\Phi}$ they have introduced the self-adjoint operators which can be interpreted as the cosine and sine operators of

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0003-4916/$ – see front matter
http://dx.doi.org/10.1016/j.aop.2013.06.013

the phase. But this really interesting result has not closed the discussion as it seems quite clear that the well defined classical phase observable should have its quantum counterpart. It is worth while to note that the problem with definition of $\hat{\Phi}$ as the self-adjoint operator canonically conjugate to $\hat{N}$ fulfilling (1) can be easily understood as a direct consequence of the celebrated Pauli theorem [5] under observation that $\hat{N}$ is bounded from below and its spectrum is discrete. Recall that the same Pauli theorem causes severe difficulties with a correct definition of the time operator as the object canonically conjugate to the Hamilton operator [6–8]. So, some methods applied to the problem of defining the time operator are analogous to the ones used in the case of searching for the quantum phase. In particular, one can look for the phase operator by performing the Weyl quantization of the classical phase of harmonic oscillator considered as a function on the phase space $\mathbb{R}^2$ [9]. However, since this function is rather involved the corresponding operator obtained from the Weyl quantization rule can reveal properties which are not pertinent to the expected properties of the correct phase operator. The similar case occurs when the classical arrival time function is quantized [7,10,11]. In 1970 Garrison and Wong [3] were able to find the self-adjoint phase operator which satisfied the commutation relation (1) on a dense subset of the Hilbert space (see also [12,13]). The problem with the Garrison–Wong phase operator $\hat{\Phi}_{GW}$ is that the probability distribution of the phase calculated for $\hat{\Phi}_{GW}$ in any eigenstate of the number operator $\hat{N}$ is not uniform [12] (see also Section 5 of the present paper). Yet, another approach to the definition of quantum phase has been considered by Popov and Yarunin [14,15] and then developed by Pegg and Barnett [16–18], and nowadays is called *the Pegg–Barnett (PB) approach*. We will study it in more detail in our paper. Here we only point out that the main idea of the PB approach is to define the phase operator in the appropriate sequence of finite-dimensional Hilbert spaces. Then all calculations concerning a given observable relevant to the phase are first accomplished in those finite-dimensional Hilbert spaces and then one takes the limit with the dimension tending to infinity. Some objections against this approach has been raised by Busch, Grabowski and Lahti [13]. Namely, they write "Nevertheless there is no reason to stick to the finite-dimensional Hilbert space: one may equally well do all calculations after performing the limit $s \to \infty$" ([13, p. 6]. Here s stands for the dimension of respective Hilbert space). In Ref. [13] the quantum phase is given by a *positive operator valued (POV) measure* (see also [19]) and this POV measure leads to the Pegg–Barnett results but without any use of finite-dimensional Hilbert spaces. It is also proved in [13] that POV measure defining the quantum phase arises from some spectral measure $E : \mathcal{B}([-\pi, \pi)) \to \mathcal{L}_+(L^2(S^1))$ by the Naimark projection $\hat{\Pi} : L^2(S^1) \to L^2(\mathbb{R}^1)$ (see Section 5 of the present paper). This result shows that the Hilbert space of states for a particle on the circle, $L^2(S^1)$, *seems to play the crucial role for understanding the quantum phase*. The same conclusion follows from a nice work by Sharatchandra [20]. The aim of our paper is to develop this idea in more detail. We intend also to show how the PB approach can be incorporated into the generalized Weyl quantization formalism. In Section 2 we introduce the *generalized Weyl quantization* and the *generalized Stratonovich–Weyl (GSW) quantizer* for a particle on the circle. In Section 3 we use the idea of the Pegg–Barnett approach to get the restricted GSW quantizer and to employ this quantizer in defining quantum observables on the cylindrical phase space $S^1 \times \mathbb{R}^1$. The results of Sections 2 and 3 enable us to find in Section 4 the angle operator with the use of GSW quantizer. We demonstrate that one can apply the Pegg–Barnett approach to rotation angle observable in a "natural way" and this leads to the sequence of angle operators in finite-dimensional Hilbert spaces quite different from the respective sequence obtained in [21]. Section 5 is devoted to the problem of incorporating the quantum phase into the generalized Weyl quantization strategy on $S^1 \times \mathbb{R}^1$. Our proposition of the solution of this problem is described by the points (1), (2) and (3) (see Section 5). As is then shown, this approach leads to the self-adjoint operator on the Hilbert space $L^2(S^1)$ which gives the same results as the POV measure approach of Refs. [13,19] and the Pegg–Barnett approach [16–18]. Moreover, the analogous strategy can be used for other physical quantities which depend on the phase ϕ and/or the number N. For example, in Section 6 we use it to find the uncertainty relation for $\hat{\Phi}$ and $\hat{N}$.

2. Generalized Stratonovich–Weyl quantizer for a particle on the circle

Let the angle coordinate on the unit circle S^1 be denoted by Θ, $-\pi \leqslant \Theta < \pi$. The Hilbert space $L^2(S^1)$ can be identified with $L([-\pi, \pi))$ or equivalently with $L^2(2\pi)$ which is the vector space