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Localised distributions and criteria for correctness in complex Langevin dynamics

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HIGHLIGHTS

- Characterisation of the equilibrium distribution sampled in complex Langevin dynamics.
- Connection between criteria for correctness and breakdown.
- Solution of the Fokker–Planck equation in the case of real noise.
- Analytical determination of support in complexified space.

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ABSTRACT

Complex Langevin dynamics can solve the sign problem appearing in numerical simulations of theories with a complex action. In order to justify the procedure, it is important to understand the properties of the real and positive distribution, which is effectively sampled during the stochastic process. In the context of a simple model, we study this distribution by solving the Fokker–Planck equation as well as by brute force and relate the results to the recently derived criteria for correctness. We demonstrate analytically that it is possible that the distribution has support in a strip in the complexified configuration space only, in which case correct results are expected.

1. Introduction

Complex Langevin (CL) dynamics [1,2] provides an approach to circumvent the sign problem in numerical simulations of lattice field theories with a complex Boltzmann weight, since it does not

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rely on importance sampling. In recent years a number of stimulating results has been obtained in the context of nonzero chemical potential, in both lower and four-dimensional field theories with a severe sign problem in the thermodynamic limit [3–8] (for two recent reviews, see e.g. Refs. [9,10]). However, as has been known since shortly after its inception, correct results are not guaranteed [11–16]. This calls for an improved understanding, relying on the combination of analytical and numerical insight. In the recent past, the important role played by the properties of the real and positive probability distribution in the complexified configuration space, which is effectively sampled during the Langevin process, has been clarified [17,18]. An important conclusion was that this distribution should be sufficiently localised in order for CL to yield valid results. Importantly, this insight has recently also led to promising results in nonabelian gauge theories, with the implementation of SL($N,\mathbb{C}$) gauge cooling [8,10].

The distribution in the complexified configuration space is a solution of the Fokker–Planck equation (FPE) associated with the CL process. However, in contrast to the case of real Langevin dynamics, no generic solutions of this FPE are known (see e.g. Ref. [19]). In fact, even in special cases only a few results are available [11,20,17,21]. In Refs. [17,18] this problem was addressed in a constructive manner by deriving a set of criteria for correctness, which have to be satisfied in order for CL to be reliable. These criteria reflect properties of the distribution and, importantly, can easily be measured numerically during a CL simulation, also in the case of multi-dimensional models and field theories [6].

A widely used toy model to understand CL is the simple integral

$$Z=\int_{-\infty}^{\infty}dx\,e^{-S},\qquad S=\frac{1}{2}\sigma x^2+\frac{1}{4}\lambda x^4, \tag{1.1}$$

where the parameters in the action are complex-valued. This model had been studied shortly after CL was introduced [22,11,23], but no complete solution was given. As we will see below, its structure, with complex σ, is relevant for the relativistic Bose gas at nonzero chemical potential [4,20]. Recently, a variant of this model (with $\sigma=0$ and λ complex) was studied by Duncan and Niedermaier [21]: in particular they constructed the solution of the FPE, using an expansion in terms of Hermite functions. They considered the case of "complex noise", in which both the real and imaginary parts of the complexified variables are subject to stochastic kicks. Unfortunately, it has been shown in the past that generically complex noise may not be a good idea, since it leads to broad distributions in the imaginary direction and hence incorrect results [17,18]. This was indeed confirmed in Ref. [21].

In this paper we aim to combine the insights that can be distilled from the criteria for correctness discussed above with the explicit solution of the FPE, adapting the method employed in Ref. [21] to the model (1.1). The paper is organised as follows. In Section 2 we discuss CL and the criteria for correctness. To keep the paper sufficiently accessible, we first briefly review how to arrive at the criteria for correctness and subsequently present numerical results, for both real and complex noises. In Section 3 we study the probability distribution in the complexified configuration space, by solving the FPE directly as well as by a brute-force construction using the CL simulation, again for complex and real noises (the latter was not considered in Ref. [21]). In Section 4 we combine our findings concerning the distribution and the criteria for correctness, and provide a complete characterisation of the dynamics. Section 5 contains the conclusion. Finally, in order to see whether the structure found numerically can be understood analytically, a perturbative analysis of the FPE is given in the Appendix.

2. Complex Langevin dynamics and criteria for correctness

We consider the partition function (1.1). We take λ real and positive, so that the integral exists, while σ is taken complex. Analytical results are available: a direct evaluation of the integral yields

$$Z=\sqrt{\frac{4\xi}{\sigma}}e^{\xi}K_{-\frac{1}{4}}(\xi), \tag{2.1}$$

where $\xi=\sigma^2/(8\lambda)$ and $K_p(\xi)$ is the modified Bessel function of the second kind. Moments $\langle x^n\rangle$ can be obtained by differentiating with respect to σ. Odd moments vanish.