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# A conformal approach for the analysis of the non-linear stability of radiation cosmologies

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ABSTRACT

The conformal Einstein equations for a trace-free (radiation) perfect fluid are derived in terms of the Levi-Civita connection of a conformally rescaled metric. These equations are used to provide a non-linear stability result for de Sitter-like trace-free (radiation) perfect fluid Friedman–Lemaître–Robertson–Walker cosmological models. The solutions thus obtained exist globally towards the future and are future geodesically complete.



## 1. Introduction

The conformal Einstein field equations have proven to be a powerful tool to analyze the stability and the global properties of vacuum, electro-vacuum and Yang–Mills-electro-vacuum spacetimes—see e.g. [1–6]. By contrast, to the best of our knowledge, there has been no attempt to make use of conformal methods to analyze similar issues in spacetimes whose matter content is given by a perfect fluid. In this article we make a first step in this direction. We discuss the stability and the global properties of a class of cosmological spacetimes having as a source a perfect fluid with trace-free energy–momentum tensor. The solutions we construct are non-linear perturbations of a Friedman–Lemaître–Robertson–Walker (FLRW) reference spacetime. The present analysis is to be regarded as a first step in the development of conformal methods for the discussion of cosmological models whose matter content is described by a perfect fluid. Hence, we restrict our attention to the simplest case from the point of view of conformal methods: *perturbations of a trace-free prefect fluid cosmological model with compact spatial sections of positive constant curvature*.

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The problem of the non-linear stability of the irrotational Euler–Einstein system for de Sitter-like spacetimes has been analyzed in [7]. This analysis shows that FLRW background solutions with pressure $\tilde{p}$ and density $\tilde{\rho}$ related by a barotropic equation of state of the form $\tilde{p} = (\gamma - 1)\tilde{\rho}$ with $1 < \gamma < \frac{4}{3}$ are future asymptotically stable under small irrotational perturbations. An extension of this analysis to the case of fluids with non-zero vorticity has been given in [8]. It is notable that the case of a pure radiation perfect fluid cannot be covered by the analysis of [7,8]. By contrast, from the point of view of conformal methods, the pure radiation perfect fluid case turns out to be one of the simplest scenarios to be considered. It should be mentioned that conformal methods have been used to pose an initial value problem for the Einstein–Euler system at the Big Bang for a class of cosmological models with isotropic singularities—see [9]. The methods used in that work do not allow, however, to obtain global existence assertions towards the future.

Our main result can be stated as follows.

**Theorem.** *Suppose one is given Cauchy initial data for the Einstein–Euler system with a de Sitter-like cosmological constant and equation of state for pure radiation. If the initial data is sufficiently close to data for a FLRW cosmological model with the same equation of state, value of the cosmological constant and spatial curvature $k = 1$, then the development exists globally towards the future, is future geodesically complete and remains close to the FLRW solution.*

A detailed and technically precise version of this result is given in Theorem 2.

**Remark 1.** Similar future global existence and stability results can be obtained using the methods of this article for a FLRW background solution with pure radiation equation of state, de Sitter-like or vanishing cosmological constant, $\lambda$, and $k = 0, -1$. These models expand indefinitely towards the future, and remarkably, their scale factor can be computed explicitly—see [10]. In the cases with $\lambda = 0$, minor technical modifications need to be introduced to account for a null conformal boundary. The stability of these models will be discussed elsewhere by means of different (conformal) methods.

**Remark 2.** The restriction of our analysis to the case of perfect fluids with trace-free energy–momentum tensor is a technical one. In the trace-free case the equation $\tilde{\nabla}^\mu \tilde{T}_{\mu\nu} = 0$ transforms homogeneously under conformal rescalings—see Eq. (25). As a result, the majority of conformal field equations for a radiation fluid can be treated with the same methods used in the analysis of the Einstein–Maxwell case [2,6]; only the equations for the fluid variables need to be analyzed in more detail, to ensure that they fit in correctly with the remainder of the system of partial differential equations (PDEs). In the case of an energy–momentum tensor with non-vanishing trace, it is an open question whether a suitable choice of conformal variables exists which leads to a regular set of conformal field equations with matter. Nevertheless, the study of the radiation fluid case is expected to give some insight towards a suitable approach for perfect fluids with different equations of state. The analysis for the wave equation in [11,12] may be a guide for this type of generalization of our analysis.

*An overview of the analysis*

In this article, we follow the conformal approach developed by H. Friedrich in a series of articles [1–3,13]. In order to help the reader following the various details and arguments we provide a brief summary of the different steps carried out in the analysis.

The key idea of our analysis is that by conformally rescaling a solution $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$ to the Einstein field equations (3) with matter described by (5), one is able to carry out a global stability analysis to the future in terms of a local analysis near the conformal boundary of the conformal extension $(\mathcal{M}, g_{\mu\nu})$ where $g_{\mu\nu} = \Theta^2 \tilde{g}_{\mu\nu}$ and $\Theta$ is a suitable conformal factor.

1. The unphysical manifold $(\mathcal{M}, g_{\mu\nu})$ does not satisfy the Einstein field equations with matter. Instead, the conformally rescaled metric $g_{\mu\nu}$ and derived fields satisfy the conformal Einstein field equations with trace-free matter (23a)–(23b) for the quantities defined in (22a)–(22h)—as first established in [2].