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General initial value problem for the nonlinear shallow water equations: Runup of long waves on sloping beaches and bays

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ABSTRACT

We formulate a new approach to solving the initial value problem of the shallow water-wave equations utilizing the famous Carrier–Greenspan transformation (Carrier and Greenspan (1957) [9]). We use a Taylor series approximation to deal with the difficulty associated with the initial conditions given on a curve in the transformed space. This extends earlier solutions to waves with near shore initial conditions, large initial velocities, and in more complex U-shaped bathymetries; and allows verification of tsunami wave inundation models in a more realistic 2-D setting.

1. Introduction

Tsunami modeling and forecast is an important scientific problem impacting coastal communities worldwide. Many models for tsunami wave propagation use the $2+1$ shallow water equations (SWE), an approximation of the Navier–Stokes equation [1]. These numerical models must be continuously verified and validated to ensure the safety of coastal communities and infrastructure [2]. Apart from verification against data from actual tsunami events, numerical models are also extensively verified against analytical solutions of the $2+1$ SWE which exist for idealized bathymetries [3]. These analytical solutions also give important qualitative insight to tsunami run-up and amplification.

Typically, the process of tsunami generation is considered as an instant vertical motion of the sea bottom ignoring the water velocities in the source. However, incorporation of the water velocities into the initial conditions is important from physical point of view, see for instance [4]. For a more complete analysis of tsunami hydrodynamics, modeling and forecast, we refer the reader to [1–3,5,6].

A classical example of an analytical solution for the $2+1$ SWE is computing the run-up of long-waves on a sloping beach [7]. Because of the symmetric bathymetry, the $2+1$ SWE reduce to the $1+1$ SWE, which could be solved directly in the physical space [8] or in the new coordinates using the Carrier–Greenspan transformation [9]. The SWE in the transformed coordinates has been extensively studied as an initial value problem (IVP) [4,10–13] and as a boundary value problem [7,14]. The IVP for waves with nonzero initial velocity have been previously derived using a Green's function in [4,10], though both solutions imply assumptions regarding the initial velocity as discussed later. Thus, the complete and exact solution to the IVP for waves with nonzero initial velocities remains a long standing open problem [4,14,15].

The $1+1$ SWE for the sloping beach have recently been generalized to model waves in sloping narrow channels using the cross-sectionally averaged $1+1$ SWE [16]. Furthermore, the hodograph transform given by [9] can be generalized to sloping bays with arbitrary cross sections, allowing a much richer problem to study [17]. Though the cross-sectionally averaged $1+1$ SWE have no analytical solution for bays with arbitrary cross sections, an analytical solution exists for symmetric U-shaped bays, i.e. bays with a cross section $z \propto y^m$ [16–18]. The known solution for sloping beaches is an asymptotic solution of such bays when $m \to \infty$.

In this letter we propose a new approach to solve the IVP for the cross-sectionally averaged $1+1$ SWE in U-shaped bays for waves with arbitrary initial velocities exactly. Our solution uses a Taylor expansion to deal with the initial data given on a curve (un-

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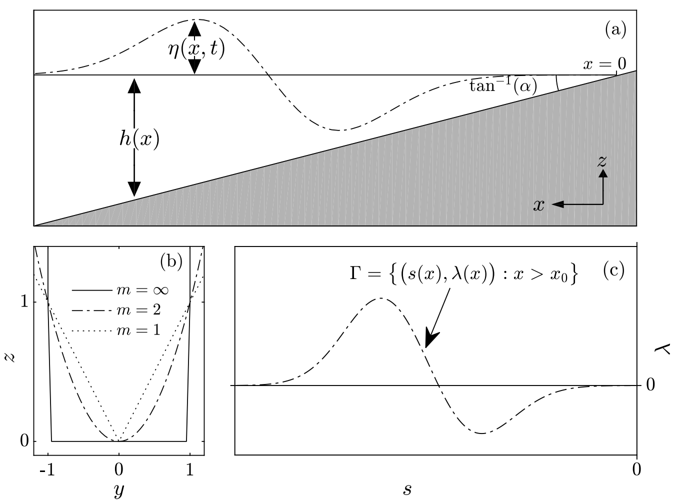

Fig. 1. Definition sketch: (a) x–z cross section with perturbed and unperturbed water heights, not to scale. (b) y–z cross section of a plane beach where $m=\infty$, a parabolic bay where $m=2$, and a V-shaped bay where $m=1$. (c) The curve Γ on which the initial conditions are prescribed in the transformed space for the wave shown in (a), not to scale.

der the Carrier–Greenspan transformation the line $t=0$ is mapped to a curve in the transformed plane), a problem that was not sufficiently treated in the previous solutions. This allows run-up computation of near shore long waves, unlike the previous IVP solutions that require the initial wave to be far from shore [4]. Additionally, we present some qualitative geophysical implications using this new solution.

2. Solution of the IVP

The cross-sectionally averaged 1 + 1 SWE for U-shaped bays describe the evolution of long waves in a sloping narrow bay with an unperturbed water height $h(x)=x$ along the main axis of the bay in dimensionless form. The wave is assumed to propagate uniformly through the bay in the x direction, a valid assumption as shown in [18,19]. The cross-sectionally averaged SWE for such bays in dimensionless form are given by [16,18] to be

$$\eta_t + u(1+\eta_x) + \beta^2(x+\eta)u_x = 0, \tag{1a}$$

$$u_t + uu_x + \eta_x = 0, \tag{1b}$$

where $u(x,t)$ and $\eta(x,t)$ are the horizontal depth-averaged velocity and free-surface elevation along the main axis of the bay, respectively, and $\beta^2 = m/(m+1)$ is the wave propagation speed along a constant depth channel. An arbitrary scaling parameter l is used to introduce the dimensionless variables $x=\tilde{x}/l$, $\eta=\tilde{\eta}/(l\alpha)$, $u=\tilde{u}/\sqrt{g\alpha l}$ and $t=\tilde{t}\sqrt{g\alpha/l}$. Here $\tilde{x},\tilde{t},\tilde{\eta}$ and $\tilde{u}$ are the dimensional variables, g is the gravitational acceleration, and α is the slope of the incline.

We use the form of the Carrier–Greenspan transformation presented in [15],

$$\varphi = u, \qquad \psi = \eta + \frac{u^2}{2} \tag{2a}$$

$$s = x+\eta, \qquad \lambda = t-u, \tag{2b}$$

to reduces (1) to the linear system

$$\Phi_\lambda + A(s)\Phi_s + B\Phi = 0, \tag{3}$$

where $\Phi(s,\lambda) = \begin{pmatrix}\varphi(s,\lambda)\\ \psi(s,\lambda)\end{pmatrix}$, $A(s) = \begin{pmatrix}0 & 1\\ \beta^2 s & 0\end{pmatrix}$, and $B = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$. This form of the Carrier–Greenspan transformation has two useful properties, the moving shoreline is fixed at $s=0$ and the resulting linear system, (3), is the linear SWE. For comparison to other texts, specifically [4,7,9,17], the transform variable $\sigma = 2\sqrt{s}/\beta$ is typically used, along with the introduction of a potential function to form a single linear second order partial differential equation.

We consider (3) with the general initial conditions in physical space $\eta(x,0)=\eta_0(x)$ and $u(x,0)=u_0(x)$. Under transformation (2b), $\eta_0(x)$ and $u_0(x)$ transform into initial conditions on a parameterized curve Γ in the (s,λ) plane, depicted in Fig. 1c, which leads to a non-trivial IVP. It is natural to parameterize this curve using the coordinate x, $\Gamma = \{\Gamma(x) : x > x_0\}$, where

$$\Gamma(x) = \big(s(x),\lambda(x)\big) = \big(x+\eta_0(x), -u_0(x)\big), \tag{4}$$

and x_0 is the x position of the shoreline at time $t=0$. The initial condition is then given by

$$\Phi|_{\Gamma(x)} = \Phi_0(x) = \begin{pmatrix} u_0(x) \\ \eta_0(x) + u_0^2(x)/2 \end{pmatrix}. \tag{5}$$

A general solution to (3) can be found using the Hankel transform to be [16,18]

$$\psi(s,\lambda) = s^{-\frac{1}{2m}} \int_0^\infty \{a(k)\cos(\beta k\lambda) + b(k)\sin(\beta k\lambda)\} \times J_{1/m}\big(2k\sqrt{s}\big)\,dk, \tag{6a}$$

$$\varphi(s,\lambda) = \frac{1}{\beta} s^{-\frac{1}{2m}-\frac{1}{2}} \int_0^\infty \{a(k)\sin(\beta k\lambda) - b(k)\cos(\beta k\lambda)\} \times J_{1/m+1}\big(2k\sqrt{s}\big)\,dk, \tag{6b}$$

where $J_\nu(\alpha)$ is the Bessel function of the first kind of order ν, and $a(k)$ and $b(k)$ are arbitrary functions determined by the initial conditions. We note that the apparent singularities at $s=0$ are removed using the asymptotic of the Bessel function of the first kind around zero.

In the piston model of generation, i.e. with zero initial velocity, the curve Γ coincides with the line $\lambda=0$. For arbitrary initial conditions on the line $\lambda=0$, using the inverse Hankel transform, we have that

$$a(k) = 2k \int_0^\infty \psi(s_*,0)\, s_*^{\frac{1}{2m}}\, J_{\frac{1}{m}}\big(2k\sqrt{s_*}\big)\,ds_*, \tag{7a}$$

$$b(k) = -2\beta k \int_0^\infty \varphi(s_*,0)\, s_*^{\frac{1}{2m}+\frac{1}{2}}\, J_{\frac{1}{m}+1}\big(2k\sqrt{s_*}\big)\,ds_*. \tag{7b}$$

For waves with zero initial velocity, using (5) and a simple change of variables, (7) simplifies to $b(k)=0$, and

$$a(k) = 2k \int_{x_0}^\infty \eta_0(x_*)\big(x_*+\eta_0(x_*)\big)^{\frac{1}{2m}} J_{\frac{1}{m}}\Big(2k\sqrt{x_*+\eta_0(x_*)}\Big) \times (1+\eta_0'(x_*))\,dx_*, \tag{8}$$

where primes denote derivatives in x. Using (6), $\varphi(s,\lambda)$ and $\psi(s,\lambda)$ can be computed. The solution is then transformed to physical space using (2). The solution over a large number of grid points can be found by interpolation using Delaunay triangulation, as in [18]. Alternatively, Newton–Raphson iterations can be used to find the solution for a particular location x or time t, as in [7,11].

If the initial wave has an initial velocity, the curve Γ may be complicated so that an exact solution does not exist. Reference [4]