Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Imaginary eigenvalues of Zakharov–Shabat problems with non-zero background

Gino Biondini *, Xudan Luo

State University of New York at Buffalo, Department of Mathematics, Buffalo, NY 14260, USA

ARTICLE INFO

Article history:
Received 30 March 2018
Received in revised form 5 June 2018
Accepted 25 June 2018
Available online xxxx
Communicated by A.P. Fordy

Keywords:
Zakharov–Shabat scattering problem
Discrete eigenvalues
Inverse scattering
Nonlinear Schrödinger equation

ABSTRACT

The focusing Zakharov–Shabat scattering problem on the infinite line with non-zero boundary conditions for the potential is studied, and sufficient conditions on the potential are identified to ensure that the problem admits only purely imaginary discrete eigenvalues. The results, which generalize previous work by Klaus and Shaw, are applicable to the study of solutions of the focusing nonlinear Schrödinger equation with non-zero background.

1. Introduction

The nonlinear Schrödinger (NLS) equation is a universal model that describes the evolution of weakly nonlinear and quasi-monochromatic wave trains in media with cubic nonlinearities. As such, it arises in many disparate physical settings such as water waves, optics, acoustics, Bose–Einstein condensation, etc. It is also known since the pioneering work of Zakharov and Shabat in 1972 [30] that the NLS equation in one spatial dimension is a completely integrable system, and as such it can be written as the compatibility condition of an overdetermined pair of linear ordinary differential equations, which are called the Lax pair. Zakharov and Shabat also showed that the initial-value problem for the NLS equation could be solved by the inverse scattering transform. Accordingly, the first half of the Lax pair for the NLS equation is referred to as the Zakharov–Shabat scattering problem, and the solution of the NLS equation plays the role of a potential there. Therefore, the study of Zakharov–Shabat scattering problems has been an ongoing area of research (e.g., see [5,17,20,27]).

Recall that the NLS equation is the compatibility condition of the matrix Lax pair

$$\mathbf{v}_x = (-i\zeta\,\sigma_3 + Q\,(x,t))\,\mathbf{v}\,, \tag{1a}$$

$$\mathbf{v}_t = (2i\zeta^2\sigma_3 + 2kQ - i\,Q_x\sigma_3 - i\,Q^2\sigma_3)\,\mathbf{v}\,, \tag{1b}$$

where $\mathbf{v}(x,t,\zeta) = (v_1, v_2)^T$, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q\,(x,t) = i\begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix} \tag{2}$$

(with σ_1 to be used later). That is, the requirement $\mathbf{v}_{xt} = \mathbf{v}_{tx}$, together with the constraint $r = \nu q^*$, yields the NLS equation,

$$iq_t + q_{xx} - 2\nu|q|^2 q = 0\,, \tag{3}$$

where $q : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$, subscripts denote partial derivatives and as usual $\nu = \mp 1$ denote the focusing and defocusing cases, respectively. Equation (1a) is referred to as the Ablowitz–Kaup–Newell–Segur scattering problem [3]. The Zakharov–Shabat scattering problem is the special case of (1) when $r(x,t) = \nu q^*(x,t)$ [with the asterisk denoting complex conjugation], in which case the compatibility condition of (1) yields precisely the NLS equation (3).

Equation (1a) can equivalently be written as the eigenvalue problem $\mathcal{L}\mathbf{v} = \zeta\mathbf{v}$ for the Dirac operator $\mathcal{L} = i\sigma_3(\partial_x - Q)$. The spectrum of the scattering problem is the set of all values of $\zeta \in \mathbb{C}$ such that nontrivial bounded eigenfunctions $\mathbf{v}(x,t,k)$ exist, and such values of ζ are referred to as the eigenvalues of the scattering problem. In particular, values $\zeta \in \mathbb{C}$ such that $\mathbf{v}(x,t,k) \in L^2(\mathbb{R})$ are referred to as the discrete eigenvalues of the problem. (As we discuss below, the above definition differs slightly from the one typically used in the development of the inverse scattering transform (IST), in which discrete eigenvalues are defined as the zeros of the analytic scattering coefficients.) The structure of the Lax pair implies that, when the potential evolves in time according to the

* Corresponding author.
E-mail address: biondini@buffalo.edu (G. Biondini).

https://doi.org/10.1016/j.physleta.2018.06.045
0375-9601/

NLS equation, the spectrum of $\mathcal{L}$ is independent of time. For this reason, we will drop the time dependence throughout this work.

In the defocusing case the Dirac operator $\mathcal{L}$ is self-adjoint [31], and therefore all eigenvalues are real. In the focusing case ($\nu = -1$), however, $\mathcal{L}$ is non-self adjoint. One can show that the reduction $r = \nu q^*$ implies that the spectrum of $\mathcal{L}$ is symmetric with respect to the real ζ-axis. It is also well known that, if the potential is even, the spectrum is also symmetric with respect to the imaginary ζ-axis. A natural question, however, is whether there exist special classes of potentials for which the spectrum possesses additional symmetries.

The above question was studied in 2002 by Klaus and Shaw [21]. Specifically, Klaus and Shaw considered a class of potentials $q(x)$ that are non-negative, smooth, L^1 functions on the real line, and such that $q(x)$ is nondecreasing for $x < 0$ and nonincreasing for $x > 0$. They were then able to show that any discrete eigenvalues ζ of (1a) are purely imaginary.

The study of nonlinear wave equations with non-zero boundary conditions (NZBC) also has a long history [17,31], and has received renewed attention in recent years (e.g., see [4,7–9,11,14,15,23] and references therein), due also in part to connections with various physical effects such as rogue waves [6,29], modulational instability [7,12,13], the dynamics of dispersive shock waves [1,2] and polarization shifts [10]. A limitation of Klaus and Shaw's result, however, is that it only applies to decaying potentials.

The properties of scattering operators with NZBC can be quite different from those of the same operators with ZBC. For example, it is well known that an "area theorem" exists for the Zakharov–Shabat operator with ZBC: no discrete eigenvalues can exist if the L^1 norm of the potential is less than $\pi/2$ [22]. (This bound, which improves the original one [3], is sharp.) However, it was recently shown that no generalization of the area theorem is possible for the same operators with NZBC, either in the focusing [7] or in the defocusing [14] case. In other words, the situation for the Zakharov–Shabat operator with NZBC is dissimilar to that for the same operators with ZBC, and is more similar instead to that for the Schrödinger operator $\mathcal{L} = -\partial^2 + q(x)$, which defines the scattering problem for the Korteweg–deVries equation [18]. On the other hand, in this work we show that the results of [21] do admit a straightforward generalization to potentials with NZBC.

To do this, we generalize the notion of "single-lobe" potentials to the case of NZBC. Specifically, we will call a single lobe potential with NZBC a function $q(x)$ which is: (i) smooth on real line, (ii) nondecreasing for $x < 0$ and nonincreasing for $x > 0$, (iii) limiting to $q(x) \to q_0$ as $x \to \pm\infty$, where $q_0 > 0$ is a constant, and (iv) $q(x) - q_0 \in L^1(\mathbb{R})$. This definition allows us to obtain the main result of this work, which is the following

Theorem 1.1. *Let $q(x)$ be a smooth, real-valued function on the real line such that*

$$q(x) \to q_0 \quad \text{as} \quad x \to \pm\infty, \qquad q(x) > q_0 \quad \text{for } x \in \mathbb{R},$$

where $q_0 > 0$ is a constant. Moreover, let $q(x) - q_0 \in L^1(\mathbb{R})$. If $q(x)$ is nondecreasing for $x < 0$ and nonincreasing for $x > 0$, any discrete eigenvalue ζ of the scattering problem (1a) is purely imaginary, and $|\zeta| > q_0$.

In section 2 we give the proof of Theorem 1.1, and in section 3 we discuss a few examples to illustrate that both of the hypotheses of the theorem (namely, constant-phase and single-lobe conditions) are indeed necessary. Section 4 ends this work with a few concluding remarks.

2. Proof of Theorem 1.1

The strategy of the proof follows that in [21], but the implementation is somewhat different due to the NZBC. First we derive some upper bounds regarding the behavior of the Jost eigenfunctions corresponding to a discrete eigenvalue. Then we derive a constraint that relates discrete eigenvalues to certain integrals of the corresponding eigenfunctions. Finally we use the bounds to establish that the real part of the discrete eigenvalue must vanish identically.

2.1. Jost eigenfunctions and upper bound estimates

Recall that in the IST for the focusing NLS equation with NZBC [8] one defines the Jost eigenfunctions as the solutions of (1a) which tend to plane wave behavior either as $x \to \infty$ or as $x \to -\infty$. In particular, for our purposes it is sufficient to introduce the columns $\phi(x,\zeta)$ and $\psi(x,\zeta)$ as

$$\phi(x,\zeta) = \begin{pmatrix} \lambda+\zeta \\ -iq_0 \end{pmatrix} e^{-i\lambda x}(1+o(1)), \quad x \to -\infty, \tag{4a}$$

$$\psi(x,\zeta) = \begin{pmatrix} -iq_0 \\ \lambda+\zeta \end{pmatrix} e^{i\lambda x}(1+o(1)), \quad x \to +\infty, \tag{4b}$$

where $\lambda(\zeta)$ is defined by the equation $\lambda^2 = \zeta^2 + q_0^2$. The set of values $\zeta \in \mathbb{C}$ such that $\lambda(\zeta) \in \mathbb{R}$ comprises the discrete spectrum of the scattering problem. In our case, this is the set $\Sigma = \mathbb{R} \cup i[-q_0, q_0]$. Without loss of generality, one can define $\lambda(\zeta)$ for all $\zeta \in \mathbb{C}$ through the analytic continuation of the principal branch of the real square root off the positive real ζ-axis with a square-root sign discontinuity across the branch cut $[-iq_o, iq_o]$. It is easy to show that, with this definition, the sign of the imaginary part of $\lambda(\zeta)$ is the same as that of ζ away from the branch cut.

The Zakharov–Shabat scattering problem possesses the usual reflection symmetry such that for every eigenvalue ζ in the upper-half plane there is a corresponding eigenvalue ζ^* in the lower-half plane [8]. Thus, without loss of generality we can restrict ourself to studying the discrete eigenvalues in the upper-half plane.

The Jost eigenfunctions (4) are rigorously defined as the solutions of suitable linear integral equations [8]. For example,

$$\phi(x,\zeta) = \begin{pmatrix} \lambda+\zeta \\ -iq_0 \end{pmatrix} e^{-i\lambda x} + \int_{-\infty}^{x} G_-(x-y,\zeta)(Q(y)-Q_0)e^{i\lambda(y-x)}\phi(y,\zeta)\,dy, \tag{5}$$

where

$$G_-(x-y,\zeta) = \frac{1}{2\lambda}[(1+e^{2i\lambda(x-y)})\lambda I - i(e^{2i\lambda(x-y)}-1)(i\zeta\sigma_3 + Q_0)], \tag{6}$$

with

$$Q_0 = \begin{pmatrix} 0 & q_0 \\ -q_0 & 0 \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$

and a similar equation for $\psi(x,\zeta)$. Using these integral equations, it was shown in [8] that, as is usually the case in the IST, both $\phi(x,\zeta)$ and $\psi(x,\zeta)$ admit analytic continuation to the upper half of the complex ζ plane.

Suppose now that $\lambda(\zeta) = \alpha + i\beta$ is a discrete eigenvalue corresponding to a certain value of ζ in the closure of the upper-half plane. It was shown in [8] that, as is usually the case in the IST, the associated Jost eigenfunctions $\phi(x,\zeta)$ and $\psi(x,\zeta)$ at this specific value of ζ are proportional each other, and that any of the $L^2(\mathbb{R})$ eigenfunctions $v(x,\zeta)$ associated to ζ are proportional to both of them. Because our definition of discrete eigenvalues require the corresponding eigenfunctions to be in $L^2(\mathbb{R})$ (as opposed to simply being bounded), it follows that $\beta = \operatorname{Im}\lambda$ must be strictly

Please cite this article in press as: G. Biondini, X. Luo, Imaginary eigenvalues of Zakharov–Shabat problems with non-zero background, Phys. Lett. A (2018), https://doi.org/10.1016/j.physleta.2018.06.045