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Finite-dimensional pseudo-bosons: A non-Hermitian version of the truncated harmonic oscillator

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ABSTRACT

We propose a deformed version of the commutation rule introduced in 1967 by Buchdahl to describe a particular model of the truncated harmonic oscillator. The rule we consider is defined on a N-dimensional Hilbert space $\mathcal{H}_N$, and produces two biorthogonal bases of $\mathcal{H}_N$ which are eigenstates of the Hamiltonians $h=\frac{1}{2}(q^2+p^2)$, and of its adjoint $h^\dagger$. Here q and p are non-Hermitian operators obeying $[q,p]=i(\mathbb{1}-Nk)$, where k is a suitable orthogonal projection operator. These eigenstates are connected by ladder operators constructed out of q, p, $q^\dagger$ and $p^\dagger$. Some examples are discussed.

1. Introduction

Quantum mechanics is often thought to be naturally associated to self-adjoint (or Hermitian[1]) operators. In particular, the dynamics is deduced out of a self-adjoint Hamiltonian, and the *observables* of the system are almost always assumed to be self-adjoint as well.

In recent years, mainly since the seminal work by Bender and Boettcher, [1], it was understood that self-adjointness is not an essential requirement, since other operators exist, not self-adjoint, having purely real (and discrete) spectra. We refer to [2–4] for some reviews on this alternative approach. What is interesting, from a mathematical point of view, is that orthonormal (o.n.) bases of eigenstates are replaced by biorthonormal sets which can be, or not, bases of the Hilbert space where the physical system lives. Also, different scalar products can play a role, and this different products produce different adjoints of the same operators. Moreover, the role of pseudospectra in connection with unbounded operators becomes relevant, [5]. Then, in a sense, loosing self-adjointness makes the mathematical structure reacher. Not only that: from a physical point of view the situation is also rather interesting since, for instance, some so-called PT-symmetric Hamiltonians can be naturally used to describe quantum systems with gain and loss phenomena, see [6,7] and references therein.

In recent years, in connection with this kind of operators, we have developed a rather general formalism based on some suitable deformations of the canonical commutation and anti-commutation relations (CCR and CAR). These deformations produce what we have called $\mathcal{D}$-pseudo bosons and pseudo-fermions. A rather complete review on both these topics can be found in [8], to which we refer for several details and for some physical applications. Later on a similar framework was proposed for quons and for generalized Heisenberg algebra, [9,10].

Here we consider a deformation of a different commutation rule, originally considered in [11], and later analyzed in [12], in connection with a truncated version of the harmonic oscillator. The operator c considered in these papers obeys the following rule

$$[c,c^\dagger]=\mathbb{1}-NK, \tag{1.1}$$

in which $N=2,3,4,\ldots$ is a fixed natural number, while K is a self-adjoint projection operator, $K=K^2=K^\dagger$, satisfying the equality $Kc=0$. The presence of the term NK in (1.1) makes it possible to find a representation of K and c in terms of $N\times N$ matrices. In fact, in absence of this term we would recover the CCR, which does not admit any finite-dimensional representation. Here, on the other hand, K, c and $c^\dagger$ act on a N-th dimensional Hilbert space, which we call $\mathcal{H}_N$. In [11] it was shown that the matrices for c and $c^\dagger$ are essentially the truncated versions of the analogous, infinite-dimensional, matrices for the bosonic annihilation and creation operators. In [11] it was also discussed how to construct an orthonormal (o.n.) basis of eigenvectors of the self-adjoint operator

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[1] We will use these two words as synonymous here.

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$H_0 = \frac{1}{2}(Q_0^2 + P_0^2)$, where $Q_0 = \frac{c+c^\dagger}{\sqrt{2}}$ and $P_0 = \frac{c-c^\dagger}{\sqrt{2}\,i}$ are the *truncated* position and momentum operators. These vectors turn out to be eigenvectors of both H_0 and K, and their explicit construction is strongly based on the fact that H_0 is a positive operator, other than being self-adjoint. This automatically imposes a lower bound on the possible eigenvalues of H_0, bound which was used in [11] to construct the set of eigenvectors. We will see that, in our extended case, positivity is apparently lost, so that we cannot adopt the same construction as in [11] for the eigenvectors of our new Hamiltonian h, constructed in analogy with H_0. Moreover, since $h \neq h^\dagger$, it is natural to analyze also what happens for $h^\dagger$, and this will produce a biorthogonal set of eigenvectors of $h^\dagger$, see Section 3, which is a basis for $\mathcal{H}_N$.

The article is organized as follows: in the next section we discuss our deformed version of the commutation rule (1.1), and we construct a set of eigenvectors for the related truncated non self-adjoint harmonic oscillator, with Hamiltonian h, see above. We call the operators a and b appearing in this deformation *finite-dimensional pseudo-bosons* (FDPBs), since they can be seen as a truncated version of the $\mathcal{D}$-PBs considered in [8]. We show explicitly how our construction works for some fixed values of N, and then we generalize the procedure to generic N. Incidentally we will find that the procedure proposed here is more explicit than that considered in [11]. In Section 3 the biorthogonal set of eigenvectors of $h^\dagger$ is deduced. We also show how these FDPBs are related to the operators c and K in (1.1). In Section 4 we discuss two examples, while our conclusions are given in Section 5.

2. Deformed commutation rules

The main object of our research is the following deformed version of the commutation rule (1.1):

$$[a, b] = \mathbb{1} - Nk. \tag{2.1}$$

Here N can be any fixed integer larger than 1, and k is an orthogonal projector: $k = k^2 = k^\dagger$. Extending what is done in [11] we also require that $ka = bk = 0$. Moreover, a and b are not, in general, one the adjoint of the other: $b \neq a^\dagger$. This is, in a sense, close to what was done in [13] first, and in [14] later, for CCR and CAR, and, in fact, what we will show here, is that we recover the same *global* functional structure (raising and lowering relations, biorthogonal sets, non-Hermitian number-like operators, ...) as in the cited papers, even if we work here in finite-dimensional Hilbert spaces of dimension not necessarily equal to 2, as we did in [14].

The first remark is that operators obeying the commutation rule in (2.1) can also be represented as matrices acting on a N-dimensional Hilbert space $\mathcal{H}_N$. This can be easily seen as follows: let S_0 be an $(N-1)\times(N-1)$ invertible matrix, and let s be a non-zero complex number. Then, if S is the diagonal block matrix with blocks S_0 and s, S^{-1} exists (but, in general, $S^{-1} \neq S^\dagger$) and, since (1.1) is implemented in $\mathcal{H}_N$, we can easily define three new matrices $a = ScS^{-1}$, $b = Sc^\dagger S^{-1}$ and $k = SKS^{-1}$. These operators, since K commutes with $S^\dagger S$, satisfy (2.1), as well as the equalities $k = k^2 = k^\dagger$ and $ka = bk = 0$. So we see that, at least in this situation, (2.1) can be represented in $\mathcal{H}_N$. Of course, other (higher-dimensional) representations could also exist. However, from now on, a, b and k will be considered as operators on $\mathcal{H}_N$.

We start our analysis by introducing two (non-Hermitian) position and momentum-like operators:

$$q = \frac{a+b}{\sqrt{2}}, \qquad p = \frac{a-b}{\sqrt{2}\,i},$$

so that $a = \frac{q+ip}{\sqrt{2}}$ and $b = \frac{q-ip}{\sqrt{2}}$. As in [11], we introduce the operator $h = \frac{1}{2}(p^2 + q^2)$. Despite of its expression, and of what happens in [11], h is not Hermitian ($h \neq h^\dagger$). Moreover, it is not even manifestly positive ($h \not\geq 0$), due to the fact that both q and p are not Hermitian. Nevertheless, we will show later in this section that the eigenvalues of h are indeed strictly positive for all possible choices of N. After few computations it is easy to deduce the following equalities:

$$[a, h] = a - \frac{1}{2}Nak, \qquad [b, h] = -b + \frac{1}{2}Nkb, \tag{2.2}$$

as well as

$$\begin{cases} h = ba + \frac{1}{2}(\mathbb{1} - Nk) = ab - \frac{1}{2}(\mathbb{1} - Nk), \\ \{a, b\} = 2h, \\ kh = hk = -\frac{1}{2}(\mathbb{1} - N)k, \end{cases} \tag{2.3}$$

which in particular imply that $[h, k] = 0$. Then we can look for common eigenstates of h and k, which we call $\varphi_{h',k'}$:

$$\begin{cases} h\varphi_{h',k'} = h'\varphi_{h',k'}, \\ k\varphi_{h',k'} = k'\varphi_{h',k'}. \end{cases} \tag{2.4}$$

Of course, since $k = k^2$, k' can only be 0 and 1. In particular, in analogy with what happens in [11], the only (possibly) non zero vector $\varphi_{h',k'}$, when $k' = 1$, is the vector with $h' = \frac{1}{2}(N-1)$, $\varphi_{\frac{1}{2}(N-1),1}$; all the other vectors, $\varphi_{h',1}$, if $h' \neq \frac{1}{2}(N-1)$, turn out to be zero. In general, the vectors $\varphi_{h',k'}$ are not mutually orthogonal in h', since $h \neq h^\dagger$, while they are orthogonal in k', since $k = k^\dagger$:

$$\left\langle \varphi_{h',k'}, \varphi_{h'',k''} \right\rangle = \left\langle \varphi_{h',k'}, \varphi_{h'',k'} \right\rangle \delta_{k',k''}. \tag{2.5}$$

It is now possible to prove that, if $a\,\varphi_{h',k'} \neq 0$, then this vector must be proportional to $\varphi_{h'-1+\frac{1}{2}Nk',0}$. This follows from the following facts: first, since $ka = 0$, $k(a\varphi_{h',k'}) = 0$. Secondly, using (2.2), we have

$$h\left(a\varphi_{h',k'}\right) = ([h, a] + ah)\,\varphi_{h',k'} = \left(h' - 1 + \frac{1}{2}Nk'\right)(a\varphi_{h',k'}).$$

Hence our claim follows. In particular we have

$$a\varphi_{h',0} = 0 \quad \Leftrightarrow \quad h' = \frac{1}{2}. \tag{2.6}$$

In fact, let us assume that $\varphi_{h',0} \neq 0$ but $a\varphi_{h',0} = 0$. Then, using (2.3), we have

$$0 = b\left(a\varphi_{h',0}\right) = \left(h - \frac{1}{2}(\mathbb{1} - Nk)\right)\varphi_{h',0} = \left(h' - \frac{1}{2}\right)\varphi_{h',0},$$

so that $h' = \frac{1}{2}$. The proof of the converse implication, i.e. that $a\varphi_{\frac{1}{2},0} = 0$, needs to be postponed but it is essentially based on the fact that $\mathcal{H}_N$ has dimension N. In fact, we will see that acting with a and b on vectors of the form $\varphi_{h',k'}$ we can produce N linearly independent (l.i.) vectors, including $\varphi_{\frac{1}{2},0}$. Their linear independence is due to the fact that they correspond to different, strictly positive, values of h' (so, even if they are not orthogonal, they are still l.i.), or to different values of k' (so they are orthogonal and, therefore, l.i., too). Then, if $a\varphi_{\frac{1}{2},0}$ is different from zero, it would be proportional to $\varphi_{-\frac{1}{2},0}$. This vector, being eigenstate of h with eigenvalue $h' = -\frac{1}{2}$ different from the other ones (see below), would be the $N+1$-th l.i. vector in a space with dimension N. This is clearly impossible. Hence (2.6) follows. Notice that, in particular, this also implies that h admits only strictly positive eigenvalues, even in absence of an manifest positivity, which was used in [11] to deduce the analogous of (2.6).

After showing that a annihilates $\varphi_{\frac{1}{2},0}$, we need now to show that b annihilates the vector $\varphi_{\frac{1}{2}(N-1),1}$: