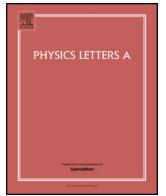




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Quantum effective potential from discarded degrees of freedom

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ABSTRACT

I obtain the quantum correction $\Delta V_{\text{eff}} = (\hbar^2/8m)[(1 - 4\xi \frac{d+1}{d})(S')^2 + 2(1 - 4\xi)S'']$ that appears in the effective potential whenever a compact d -dimensional subspace (of volume $\propto \exp[\mathcal{S}(x)]$) is discarded from the configuration space of a nonrelativistic particle of mass m and curvature coupling parameter ξ . This correction gives rise to a force $-\langle \Delta V'_{\text{eff}} \rangle$ that pushes the expectation value $\langle x \rangle$ off its classical trajectory. Because ΔV_{eff} does not depend on the details of the discarded subspace, these results constitute a generic model of the quantum effect of discarded variables with maximum entropy/information capacity $\mathcal{S}(x)$.

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1. Introduction

It is often possible and desirable to ignore specific degrees of freedom of a system, and focus on those that remain. For example, consider a nonrelativistic particle in a curved two-dimensional space

$$ds^2 = dx^2 + [b(x)]^2 d\phi^2, \quad (x, \phi) \in \mathbb{R} \times [0, 2\pi), \quad (1)$$

as illustrated in Fig. 1. If the particle also encounters a potential $V_0(x)$ then its action is

$$\mathcal{I}[x(t), \phi(t)] = \int dt \left[\frac{m}{2} (\dot{x}^2 + b^2 \dot{\phi}^2) - V_0 \right], \quad (2)$$

giving rise to the following equations of motion:

$$m\ddot{x} = mb'b\dot{\phi}^2 - V'_0, \quad (3)$$

$$mb^2\dot{\phi} = p_\phi = \text{const.} \quad (4)$$

Now suppose we only wish to describe the behaviour of the x coordinate of this particle – perhaps ϕ is unobservable in practice, or happens to be irrelevant to whatever applications we have in mind. At the classical level, we can separate the x -motion from the ϕ -motion as follows. Let us write the action (2) as

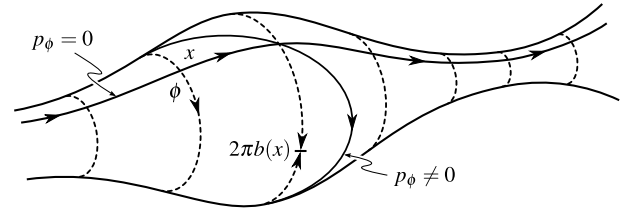


Fig. 1. As classical particles move over the curved space (1) their x coordinate can be predicted without reference to ϕ , using the reduced action (7). However, quantum particles encounter an addition potential ΔV_{eff} due to variations in the physical size $2\pi b$ of the discarded subspace $\phi \in [0, 2\pi)$.

$$\mathcal{I}[x(t), \phi(t)] = \int dt \left[\frac{m}{2} \dot{x}^2 + \frac{1}{2mb^2} (mb^2 \dot{\phi} - p_\phi)^2 + \dot{\phi} p_\phi - \frac{p_\phi^2}{2mb^2} - V_0 \right], \quad (5)$$

and note that

$$\begin{aligned} & \frac{\delta}{\delta x(t)} \int dt \left[\frac{1}{2mb^2} (mb^2 \dot{\phi} - p_\phi)^2 + \dot{\phi} p_\phi \right] \\ &= -\frac{b'}{mb^3} (mb^2 \dot{\phi} - p_\phi)^2 + \frac{2b'\dot{\phi}}{b} (mb^2 \dot{\phi} - p_\phi), \end{aligned} \quad (6)$$

which vanishes on the $\delta/\delta\phi$ equation of motion (4). If we only want to determine $x(t)$, we can therefore discard the second and third terms in (5) and work with the reduced action:

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$$\mathcal{I}[x(t)] \equiv \int dt \left[\frac{m}{2} \dot{x}^2 - V_{\text{cl}} \right], \tag{7}$$

where the (classical) effective potential is

$$V_{\text{cl}} = V_0 + \frac{p_\phi^2}{2mb^2}. \tag{8}$$

The reduced action (7) generates the correct equations of motion for x , consistent with substituting (4) into (3), and allows us to treat the particle as though it were living in a reduced configuration space

$$ds^2 = dx^2, \quad x \in \mathbb{R}. \tag{9}$$

We no longer need to refer to ϕ , and can think of p_ϕ as a parameter of the system. For a concrete application of this formalism, recall Newtonian orbital mechanics: with $b(x) = x$, the metric (1) describes a flat plane with radial coordinate x , and $V_{\text{cl}} = V_0 + p_\phi^2/2mx^2$ is the standard centrifugal potential.

It is important to realise, however, that once quantum effects are considered, the above procedure is no longer valid. If we naïvely quantize the reduced system (7) we will not arrive at the correct result: that obtained by quantizing the original system (2) and then reducing its configuration space. As we will see, the correct result differs from the naïve one by a quantum correction to the effective potential ΔV_{eff} , dependent on the physical size $\text{Vol}_\phi = 2\pi b(x)$ of the discarded subspace $\mathcal{M}_\phi \cong [0, 2\pi)$.

2. Discarding a single variable

Let us work in the Schrödinger picture, and first confirm the existence of ΔV_{eff} for the simple system above. As usual, we describe the quantum particle with a wavefunction, a scalar field $\Psi(x, \phi, t)$ that defines coordinate-invariant probabilities via integrals of the form

$$P = \int dx d\phi \sqrt{g} |\Psi|^2, \tag{10}$$

where $\sqrt{g} \equiv \sqrt{\det(g_{ij})}$ is the covariant measure endowed by the metric g_{ij} . [(1) $\Rightarrow \sqrt{g} = b$.] In curved space, Ψ obeys the covariant Schrödinger equation¹:

$$i\hbar \partial_t \Psi = \left[\frac{\hbar^2}{2m} (-\nabla^2 + \xi R) + V_0 \right] \Psi, \tag{11}$$

where the Laplacian

$$\nabla^2 = \frac{1}{\sqrt{g}} \partial_i \sqrt{g} g^{ij} \partial_j \tag{12}$$

and the Ricci scalar $R \equiv R_{ij} g^{ij} \equiv R^k{}_{ikj} g^{ij}$ are constructed from the metric g_{ij} .² [(1) $\Rightarrow R = -2b''/b$.] The form of (11) is fixed by coordinate invariance, unitarity, locality, dimensional considerations, and the limits $R \rightarrow 0$, $V_0 \rightarrow 0$; however, the curvature coupling parameter $\xi \in \mathbb{R}$ is an arbitrary dimensionless constant, representing a quantization ambiguity of the system [1–3]. One can choose

¹ The first systematic treatment of quantum mechanics in curved space is due to DeWitt [1] whose paper includes a canonical derivation of the covariant Schrödinger equation. For another perspective on the origin and ambiguity of the curvature term, see [2]. A more modern approach can be found in [3].

² The curved tube (1) is the *entire* configuration space of the system, so the covariant Schrödinger equation (11) can refer only to the *intrinsic* geometry of this manifold. Contrast this with a particle that actually exists in \mathbb{R}^3 , but is constrained to a two-dimensional surface $\Sigma \subset \mathbb{R}^3$ by a steep potential well: here, the extrinsic curvature of Σ will also play a role [4–8].

to invoke ‘minimal coupling’ $\xi = 0$, or motivate a conformal coupling according to some other principle or consideration [9,10]. For the sake of generality, we leave ξ unspecified.

Having quantized the original system, we proceed to discard the ϕ subspace. In order to make p_ϕ a parameter of the system, we must first insist that the particle be in an eigenstate of the angular momentum operator:

$$\hat{p}_\phi \Psi \equiv -i\hbar \partial_\phi \Psi = p_\phi \Psi. \tag{13}$$

This requirement serves as the analogue of equation (4) and ensures that the particle’s ϕ behaviour is sufficiently simple that the x dynamics can be described in isolation. The states of interest are then

$$\Psi = \frac{e^{ik\phi}}{\sqrt{2\pi b(x)}} \Psi_x(x, t), \quad p_\phi/\hbar = k \in \mathbb{Z}, \tag{14}$$

where the normalisation of Ψ_x ensures that probabilities (10) become integrals of the form

$$P = \int dx |\Psi_x|^2, \tag{15}$$

without any reference to the ϕ subspace. Hence we can think of Ψ_x as the wavefunction of the particle on the reduced configuration space (9).

To obtain the evolution equation for Ψ_x , we simply insert (14) into (11). We arrive at a reduced Schrödinger equation

$$i\hbar \partial_t \Psi_x = \left[-\frac{\hbar^2}{2m} \partial_x^2 + V_{\text{qu}} \right] \Psi_x, \tag{16}$$

where the *quantum* effective potential

$$V_{\text{qu}} \equiv V_{\text{cl}} + \Delta V_{\text{eff}} \tag{17}$$

has an additional contribution

$$\Delta V_{\text{eff}} = \frac{\hbar^2}{2m} \left[-\frac{1}{4} \left(\frac{b'}{b} \right)^2 + \frac{1-4\xi}{2} \left(\frac{b''}{b} \right) \right], \tag{18}$$

as promised. There are a few things to note about this quantum correction. First, this effect is not purely a consequence of spatial curvature: even for the flat case $b = x$ we find $\Delta V_{\text{eff}} \neq 0$. Second, ΔV_{eff} cannot be made to vanish identically by some choice of ξ . Third, ΔV_{eff} does not depend on p_ϕ , so all states (14) experience the same correction.

To illustrate an important physical implication of ΔV_{eff} , we now consider an arbitrary state:

$$\Psi = \sum_{k=-\infty}^{\infty} \frac{e^{ik\phi}}{\sqrt{2\pi b(x)}} \Psi_x^k(x, t), \tag{19}$$

where $\int dx \sum_k |\Psi_x^k|^2 = 1$ ensures proper normalisation. The expectation value of a function $f(x, p_\phi)$ is then

$$\begin{aligned} \langle f(x, p_\phi) \rangle &\equiv \int dx d\phi \sqrt{g} \Psi^* f(x, -i\hbar \partial_\phi) \Psi \\ &= \int dx \sum_{k=-\infty}^{\infty} |\Psi_x^k|^2 f(x, \hbar k). \end{aligned} \tag{20}$$

As each Ψ_x^k obeys the reduced Schrödinger equation (16) with $p_\phi = \hbar k$, it follows that $\langle x \rangle$ evolves according to

$$m \partial_t^2 \langle x \rangle = -\langle V'_{\text{cl}} + \Delta V'_{\text{eff}} \rangle. \tag{21}$$

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