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1-D harmonic oscillator in MONDified inertia

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ABSTRACT

In this paper we study the dynamics of a harmonic oscillator with laws of motion prescribed by MOND (Modified Newtonian Dynamics) in its modified inertia formulation. A differential equation for a 1D harmonic oscillator is obtained and several features of its solution are analyzed. Particular attention is given to the deep MOND limit regime, where the equations of motion are significantly different from the Newtonian one.

1. Introduction

MOND is a kind of dynamics proposed in the '80s by Milgrom [1], [2]. This modification of Newtonian dynamics was proposed to fit galaxies rotation curves without using dark matter [3], [4].

A lot of work was done on dynamics of systems subjects to gravity like stars and galaxies [5], [6], [7]. The best prediction of MOND theory concern the physics of galaxies [8], for example the Tully–Fisher and Faber–Jackson relations are in good agreement with the MOND paradigm. On the other hand for cluster of galaxies MOND doesn't explain completely the mass discrepancy.

MOND is fundamentally divided in two formulations: modified gravity (MG) and modified inertia (MI). Modified gravity involves only a modification of the gravitational potential, while modified inertia is a modification of all the forces. So in modified inertia systems subjects to any kind of forces have a modification of their dynamics. MOND was constructed to recover Newtonian dynamics when $a \gg a_0$ where a is the acceleration of the system and a_0 a constant with the dimension of an acceleration. When $a \ll a_0$ the system is in the so called deep MOND limit (DML). This indicates that is the acceleration which discriminates between Newtonian and MOND dynamics. The commonly accepted value of a_0 is $a_0 \approx 1.2 \times 10^{-10}$ m/s^2 and has been obtained by a large amount of physical observations. The most complete and recent survey is [9].

In this paper we want to study a one dimensional harmonic oscillator with MOND dynamics. Obviously a harmonic oscillator is a system which is more reproducible in comparison to galaxy. So if a modification of inertia is really necessary the dynamics of a harmonic oscillator could be a good benchmark.

The modified inertia paradigm is a modification of the Newtonian equation of the form:

$$\vec{F} = m\mu\left(\frac{|\vec{a}|}{a_0}\right)\vec{a} \tag{1}$$

where $\mu(x)$ is called interpolating function. This connects the Newtonian regime to the DML one, $\mu(x)$ is a continuous function. In order to do this interpolation the $\mu(x)$ has to satisfy the following relation:

$$\mu(x) = \begin{cases} 1 & \text{if} \quad |x| \gg 1 \\ x & \text{if} \quad |x| \ll 1. \end{cases} \tag{2}$$

Looking back to the (1) we have that for accelerations much larger than a_0 the Newtonian dynamics is recovered. For accelerations much smaller than a_0 we get the DML. In this limit the force law (1), in one dimension, becomes:

$$F = m\frac{a^2}{a_0}\text{sgn}(a) \tag{3}$$

where sgn is the sign function.

Actually a more general treatment of MI is based on a modification of the kinetic part of the action $S_k[\vec{r}(t), a_0]$. In this contest the kinetic action is a functional of the whole trajectory, and function of the constant a_0. An action of this kind leads to different conserved quantities and adiabatic invariants with respect to MG formulation [7]. It is also possible to construct a theory in MI without *external field effect* (EFE) [10]. We'll talk more about EFE in the discussion of the results. Another fundamental property of such a

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theory is the non locality in time (under the requirement of Galilei invariance) [11], [12].

2. Harmonic oscillator equation

For the harmonic force in 1-D we have:

$$F_h = -kx \tag{4}$$

where k is a positive constant which is related to the angular velocity ω of oscillation by the relation $k = m\omega^2$.

For our purpose of calculation we use the force modification law prescription. So equating (1) and (4) we obtain the differential equation for the harmonic oscillator with modified inertia. For a more general treatment we define some new variables in order to have an adimensional equation. We define: $y \equiv \frac{x}{x_0}$ where x_0 is the maximal amplitude (initial deviation); $\tau \equiv \omega t$ and $\xi \equiv \frac{\omega^2 x_0}{a_0}$. So the equation for the MOND harmonic oscillator reads:

$$\mu(\xi|\ddot{y}|)\ddot{y} = -y. \tag{5}$$

Equation (5) depends on the parameter ξ. Remembering that it is defined as $\xi = \frac{\omega^2 x_0}{a_0}$, it can be thought as a parameter which indicates the average acceleration of the system in units of a_0. Using equation (3), we obtain the general equation for the harmonic oscillator in DML:

$$\xi\ \ddot{y}^2 \mathrm{sgn}(\ddot{y}) = -y. \tag{6}$$

2.1. More on deep MOND limit

Looking at equation (5), we note that it is the argument of the function μ the element which controls the regime of motion. If the argument is much greater than 1 then the equation becomes the same obtained with Newtonian law. While if the argument is much smaller than 1 the equation becomes the (6). Now we want to see when the DML occur. The arguments of μ in eq. (5) are ξ and $|\ddot{y}|$, so there can be two possibilities.

- $\xi \ll 1$, so the typical accelerations of the system are always lower than a_0. Therefore the system is in the DML for all time and also $|\ddot{y}|$ is lower than 1.
- $|\ddot{y}| \ll 1$ but $\xi > 1$. This situation occurs for every system, because there is always, though small, a range of space where the acceleration is lower than a_0. This is easy to check: just look at the Newtonian equation for harmonic oscillator: $\ddot{x} = \omega^2 x$. It's trivial that for enough small x, the acceleration $\ddot{x}$ can be smaller than a_0.

It can be demonstrated that in the DML there exist a whole family of solutions for the equation of motion [13]. This family of solution depends on the particular form of the potential. For the harmonic oscillator the family of solutions has the form:

$$y_\alpha = \alpha^4 y(\tau/\alpha) \tag{7}$$

with α a real parameter.

Now we prove that (7) is actually a solution of equation (6). Let's start by inserting the expression for y_α in (6):

$$\xi\ \left[\alpha^4 \frac{d^2}{d\tau^2} y\left(\frac{\tau}{\alpha}\right)\right]^2 = -\alpha^4 y\left(\frac{\tau}{\alpha}\right)$$
$$\xi\ \left[\alpha^2 \ddot{y}\left(\frac{\tau}{\alpha}\right)\right]^2 = -\alpha^4 y\left(\frac{\tau}{\alpha}\right)$$
$$\xi\ \alpha^4 \ddot{y}^2\left(\frac{\tau}{\alpha}\right) = -\alpha^4 y\left(\frac{\tau}{\alpha}\right)$$
$$\Rightarrow \xi\ \ddot{y}^2\left(\frac{\tau}{\alpha}\right) = -y\left(\frac{\tau}{\alpha}\right) \tag{8}$$

which is equal to (6). We have supposed $\mathrm{sgn}(\ddot{y}) = 1$, this does not affect the result. For gravitational potential the family of solutions in DML leads to scale invariance for velocity, in accordance with the constant velocity of stars at the edge of the galaxies (more properly when accelerations are lower than a_0). This is not the case for the harmonic oscillator where velocity is not scale invariant as it can be seen easily from equation (7).

3. Analysis and manipulation of equations

Generally equation (5) can not be solved explicitly. The interpolating function makes the differential equation *non linear* unlike the Newtonian one that is linear. This non linearity leads to the following feature: if y and $\tilde{y}$ are solution of (5) then $\tilde{\tilde{y}} = y + \tilde{y}$ is not solution. In other words we cannot superimpose two (or more) different solutions.

We will now look for solutions for the differential equation with initial conditions: $y(0) = 1$ and $\dot{y}(0) = 0$. The condition $y(0) = 1$ is automatically obtained remembering the definition of y: $y = x/x_0$. In fact at the time $t = 0$, x is equal to x_0, so y is equal to one. The second condition has been chosen for simplicity. The Newtonian equation for the harmonic oscillator reads:

$$\ddot{y} = -y \tag{9}$$

with solution:

$$y(\tau) = \cos(\tau) \tag{10}$$

To go further we have to choose a specific interpolating function. Two of the most used function are the *simple interpolating function* and *standard interpolating function*.

$$\mu(x) = \frac{x}{1+x} \text{ simple interpolating function,} \tag{11}$$

$$\mu(x) = \sqrt{\frac{x^2}{1+x^2}} \text{ standard interpolating function.} \tag{12}$$

Using these expressions we can find the associated differential equations. With the *simple interpolating function* (11) in equation (5) we get:

$$\frac{\xi\ \ddot{y}^2\ \mathrm{sgn}(\ddot{y})}{1+\xi|\ddot{y}|} + y = 0. \tag{13}$$

While putting the *standard interpolating function* (12) in (5) we obtain:

$$\sqrt{\frac{\xi^2\ \ddot{y}^2}{1+\xi^2\ddot{y}^2}}\ \ddot{y} + y = 0. \tag{14}$$

Equation (13) is easier to handle. However (13) in that form is not useful. Let's perform some steps to bring (13) in a more clear version. Note that $\mathrm{sgn}(\ddot{y}) = -\mathrm{sgn}(y)$, so (13) becomes:

$$-\frac{\xi\ \ddot{y}^2\ \mathrm{sgn}(y)}{1+\xi|\ddot{y}|} + y = 0. \tag{15}$$

Now for $y > 0$ eq.(15) can be rewritten as:

$$-\xi\ddot{y}^2 - \xi\ddot{y}\ y + y = 0. \tag{16}$$

Basically we have to solve a second degree equation:

$$-\xi x^2 - \xi x\ y + y = 0 \tag{17}$$

where we have replaced $\ddot{y}$ with x. Eq. (17) has the two solutions: