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An efficient series approximation for the Lévy α -stable symmetric distribution

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ABSTRACT

A relevant problem in the statistical and mathematical physics literature is to derive numerically accurate expressions to calculate Lévy α -stable distributions $P_\alpha(x; \beta)$. On the formal side, important exact results rely on special functions, such as Meijer-G, Fox-H and finite sums of hypergeometric functions, with only a few exceptional cases expressed in terms of elementary functions. Hence, from a more practical point of view, methods such as series expansions are in order, e.g., to allow for the estimation of the Lévy distribution with high numerical precision, even though most of the existing approaches are restricted to a subset of the distribution parameters and/or usually demand relatively time-consuming sophisticated algorithms. Here we present a rather simple truncated expansion for the case of symmetric Lévy distributions $P_\alpha(x)$ ($\beta = 0$). This is achieved by dividing the full range of integration into windows, performing proper series expansion inside each, and then calculating the integrals term by term. The obtained representation is convergent for any $0 < \alpha \leq 2$. Moreover, its accuracy is directly controlled by the number of terms in the truncated expression, being straightforward to implement numerically. As we show with different examples, for almost all allowable α 's the calculations lead to $P_\alpha(x)$ with reasonable low absolute error for computationally inexpensive simulations.

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1. Introduction

Stochastic analysis methods are useful to describe a large number of phenomena in natural and social systems, belonging to diverse areas, including physics, biology, economics, and sociology. A fundamental result in this context is the central limit theorem (CLT) [1–4]. The generalized CLT (GCLT), due to Paul Lévy and others, has even broader implications, dealing with a whole family of stable distributions [5–9]. In the ‘classical’ case, a weak version of the CLT establishes that the sum of independent and identically distributed random variables of finite variance converges to a Gaussian distribution. The GCLT relaxes this condition of finite variance, but instead the sum converges to the so-called Lévy α -stable distribution, which is parameterized by the Lévy index $0 < \alpha \leq 2$ [10–17].

In the case of finite variance, the underlying dynamics is associated with processes presenting normal diffusion, with Gaussian propagators and mean square displacement of the particles scaling linearly with time. In contrast, a diverging variance is related to anomalous diffusion. In this case, the corresponding dynamics very often display long-range correlations, special kinds of disorder and randomness, and highly non-linear behavior. In this context, Lévy stable distributions have been used to model many-particle quantum problems [18], random lasers [19] and laser cooling [20], solid polymers [21], financial data [22], and distinct aspects of thermodynamics [23,24], to cite just few examples. It has also important links to areas such as fractional calculus, relevant in topics such as viscoelastic, anomalous diffusion, and heat transfer [25, 26].

We can represent all the possible stable distributions [1] through a Fourier transform. The α -stable Lévy distribution has the probability density function $f(x; \alpha, \beta, \mu, \sigma)$ given by

$$f(x; \alpha, \beta, \mu, \sigma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \exp[\phi(t)] \exp[-ixt], \quad (1)$$

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where

$$\phi(t) = \begin{cases} it\mu - |\sigma t|^\alpha \left(1 - i\beta \operatorname{sign}[t] \tan\left[\frac{\pi}{2}\alpha\right]\right) & \text{for } \alpha \neq 1, \\ it\mu - |\sigma t| \left(1 + i\beta \frac{2}{\pi} \operatorname{sign}[t] \ln[|t|]\right) & \text{for } \alpha = 1. \end{cases}$$

Here μ, σ are reals and $\beta \in [-1, 1]$. The Lévy index $\alpha \in (0, 2]$ governs the asymptotic behavior of the distribution $f(x; \alpha, \beta, \mu, \sigma)$ in the form of the power-law tail $\sim x^{-\alpha-1}$. Hence, the Lévy index parametrizes the type of statistics related to the random variable. For example, when there are large (nonetheless uncorrelated) velocity fluctuations with diverging second moments, the particle displacement is given by a Lévy stable distribution with $0 < \alpha < 2$. For $\alpha = 2$, one recovers the Gaussian, since the second moment is finite and the usual CLT is valid. The other parameters describe the asymmetry or skewness $\beta \in [-1, 1]$, the shift or location ($\mu \in (-\infty, +\infty)$), and the scale ($\sigma \in (0, \infty)$).

Exact formal expressions for the Lévy α -stable distributions, Eq. (1), exist in terms of the involving Meijer-G and Fox-H functions, which are generally expressed in the form of Mellin-Barnes integrals (see, e.g., [27–31]). But due to the practical importance of Lévy distributions in a huge diversity of phenomena and the fact that only in few cases (e.g., $\alpha = 1$ and $\beta = 0$ Cauchy, and the boundary limit $\alpha = 2$ Gaussian) Eq. (1) can be cast in terms of elementary functions (simple to estimate), many efforts have been made to devise alternative protocols to calculate (either exactly or approximately) Eq. (1). In this way, some series expansions for the Lévy distribution have been proposed [32–35] (mainly in the symmetric $\beta = 0$ case). For instance, in a classical work, Bergström [34] have addressed exact series representations for Lévy α -stable distributions, deriving two distinct convergent expressions: one, B1, for $0 < \alpha < 1$ (curiously, with B1 also leading to the correct asymptotical limit, $|x|$ large, when $1 \leq \alpha < 2$); and other, B2, for $1 < \alpha \leq 2$. As it concerns numerical applications, the most usually considered is B2 (see, e.g., [35]). Other approaches (relying on both special functions – as finite sums of hypergeometric functions – and expansions) [36–41] include expressions valid for particular cases of the distribution parameters, e.g., rational α 's.

In this contribution we propose a different approach to the problem. We deduce a unique (truncated) series expression for Lévy distributions valid for any $0 < \alpha \leq 2$, yielding fairly good numerical convergence. Also, our formula is easy to implement in any numerical coding platform. We concentrate on the important general case of the symmetric ($\beta = 0$) Lévy α -stable distribution with $0 < \alpha \leq 2$.

This paper is organized as follows. In Section 2 we describe our method, which basically consists in splitting the full integration range into regular intervals (piecewise integration). In each interval (or window) we approximate the integrand by a proper expression allowing analytical integration. The final form for the symmetric Lévy α -stable distribution for any $0 < \alpha \leq 2$ is hence a sum over all the (relevant) windows. Examples illustrating the good numerical convergence are presented. Final remarks and conclusion are drawn in Section 3.

2. Method and results

We address the symmetric class of distributions, i.e., $\beta = 0$. In this case one can always perform for Eq. (1) the rescaling $\mu/\sigma \rightarrow \mu$, $x/\sigma \rightarrow x$, $\sigma f \rightarrow f$, and the variable translation, $x - \mu \rightarrow x$. Then, without loss of generality we likewise can set $\sigma = 1$ and $\mu = 0$ in Eq. (1). From these considerations, $f(x; \alpha, 0, 0, 1) \equiv P_\alpha(x)$ is readily written as a cosine Fourier transform (with $0 \leq x < \infty$ and assuming $P_\alpha(-x) = P_\alpha(x)$)

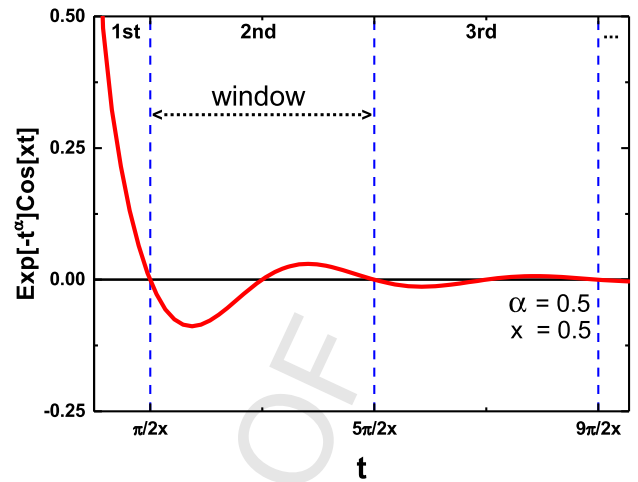


Fig. 1. We divide the Fourier integral, Eq. (2), into non-overlapping windows for piecewise integration. The dashed lines indicate the edges of the windows.

$$P_\alpha(x) = \frac{1}{\pi} \int_0^\infty dt \exp[-t^\alpha] \cos[xt]. \quad (2)$$

It is not difficult to show that $P_\alpha(x)$ is a monotonically decreasing function of x and that for $x = 0$, the above integral leads to the exact result (for $\Gamma[\cdot]$ the Gamma function)

$$P_\alpha(0) = \Gamma[1/\alpha]/(\pi\alpha), \quad (3)$$

with $P_\alpha(0) \rightarrow \infty$ for $\alpha \rightarrow 0$.

In trying to obtain an approximation expression for $P_\alpha(x)$, a first idea might be to expand the exponential in Eq. (2), so that terms of the form $(t - t_0)^m \cos[xt]$ would be relatively straightforward to calculate. But then, the issue is to properly choose the number of series terms $m_{\max} = M$ and the expansion center t_0 , allowing a reasonable good control of the numerical error regardless of the α and x considered. Certainly, a constant t_0 for any x and α would not be a convenient choice.

The hint to solve this problem is to subdivide the variable t full integration domain into intervals or 'windows'. The first (interval 0) of length $\pi/(2x)$ corresponds to the range $0 \leq t < \pi/(2x)$. The subsequent intervals, of length $2\pi/x$ each and labeled as $\mathcal{I}_n$ (for $n = 1, 2, \dots$), correspond to $(\pi/2x)(4n - 3) \leq t < (\pi/2x)(4n + 1)$. We notice, in particular, that the $\cos[xt]$ function is odd about the center $t_n = (\pi/2x)(4n - 1)$ of the intervals $\mathcal{I}_n$, furthermore vanishing at their extremes, see Fig. 1.

In this subdivision scheme is natural to treat the first interval separately from the others. Actually, as we are going to see below, Eq. (2) can be written as a sum of two convergent series, one solely for the interval 0 and the other for all the remaining $\mathcal{I}_n$'s. We shall leave to the end of this section, given x and α , a discussion about the numerical accuracy of the final (truncated) series for $P_\alpha(x)$ in terms of the number of terms used in the expansion.

We start by considering the first integration window, $0 \leq t < \pi/(2x)$, in Eq. (2). Let us define

$$g_\alpha(x) = \frac{1}{\pi} \int_0^{\pi/2x} dt \exp[-t^\alpha] \cos[xt], \quad (4)$$

and make the substitution $u = t^\alpha$, to get

$$g_\alpha(x) = \frac{1}{\alpha\pi} \int_0^{(\pi/2x)^\alpha} du \exp[-u] \cos[xu^{1/\alpha}] u^{1/\alpha-1}. \quad (5)$$

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