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Quantum state nonclassicality in weak measurements in view of Bochner's criteria

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ABSTRACT

We introduce a new formulation of nonclassicality in weak measurements based on probabilistic behavior of "quasi-moments" of a weakly measured observable. New definition determines existence of classical probabilistic interpretation and can be applied equally to quantum systems without classical counterpart in the usual sense. We show that the only consistent approach to define classicality in weak measurements should be based on the proper behavior of "quasi-moments" according to Bochner's theorem.

1. Introduction

The concept of nonclassicality in weak measurements has been introduced in Ref. [1]. The main idea is based on two observations. First, classical models for weak measurements with postselections show that weak values are nothing but the average of the measured dynamical variable weighted by a conditional distribution [2]. Therefore, for example, any positive valued dynamical variable must have positive weak value. Second, there exist quantum states for which weak values of a positive observable behave anomalously and take negative values. This means that regardless of the model chosen, no classical-like description of the weak measurement exists. These states are known as nonclassical states in the sense of weak measurements. The authors of Ref. [1]relate this nonclassicality to non-positivity of Margenau–Hill distributions and their corresponding conditional quasidistributions. In all examples that have been introduced in Refs. [1,3,4] there exists such a coexistence between nonclassical weak measurements and non-positive Margenau–Hill distribution.

In this paper we introduce a more generalized definition of nonclassicality in weak measurements. Instead of search for a phase space model of the weak measurement, or to prove its impossibility, we focus on "quasi-moments" and associated conditional quasiprobability distribution. We use Bochner's theorem to obtain a necessary and sufficient hierarchy of conditions on quasi-moments to be moments of a bona fide probability distribution. We show that the positivity of a weak value of any positive valued dynamical variable corresponds only to one of the Bochner's conditions. If any other such condition fails to represent a legitimate probability distribution function, the quantum state will be considered as nonclassical in weak measurements. From this point of view, in general, it is not possible to attribute this nonclassicality to non-positivity of Margenau–Hill distribution. We show that the observable that its weak measurements is in order and the postselected eigenstates uniquely determine a quasidistribution, usually not the phase space one, to a conditional quasidistribution of which we can associate the quasi-moments.

This paper is organized as follows. In Sec. 2 we review the Glauber nonclassicality and the problem of moments and characteristics of quasi-marginal distributions. In addition, the role of Bochner's criteria for nonclassicality has been analyzed. Sec. 3 is devoted to a brief review of weak measurements with postselection and its classical counterpart. The aim of this review is to obtain classical constrains on the moments of a dynamical variable whose weak measurement is in order. In Sec. 4 we show that, in general, one cannot attribute the nonclassicality in weak measurements to non-positivity of the Margenau–Hill distribution. In Sec. 5 we introduce a new definition for nonclassicality in weak measurements based on the behavior of quasi-moments of a conditional quasidistribution related to weak measurements. Through some examples, we highlight the similarities and difference with previous works [1,3,4]. In Sec. 6 we discuss the relation between

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Glauber nonclassicality and nonclassicality in weak measurements. The last section is devoted to conclusions.

2. Quasi-marginal distributions and Glauber classicality criterion

Density operator of a single mode quantized radiation field can be expanded diagonally in terms of coherent states [5,6]

$$\hat{\rho} = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|, \tag{1}$$

where $P(\alpha)$ is the Glauber–Sudarshan representation of the quantum state. For a normally ordered observable $\hat{s} =: s\left(\hat{a}, \hat{a}^\dagger\right):$ with c-number representation $\langle\alpha|\hat{s}|\alpha\rangle$, the optical equivalence theorem

$$\langle\hat{s}\rangle = \int d^2\alpha P(\alpha) s\left(\alpha, \alpha^*\right),$$

is a classical-like relation for expectation values. Therefore, as long as we are interested in normally ordered operators, $P(\alpha)$ resembles the classical phase space distributions. If $P(\alpha)$ is non-positive, the state is considered as nonclassical [7,8], called P-nonclassical from now on.

Clearly, in general $\langle\alpha|\hat{s}^n|\alpha\rangle \neq \langle\alpha\left|\hat{s}\right|\alpha\rangle^n$ and thus, the equivalence theorem fails to resemble classical statistical relations for the calculation of $\langle\hat{s}^n\rangle$ even if the state is P-classical. In fact, experimental procedures lead to measurement of sequence of quasi-moments $\{\langle:\hat{s}^n:\rangle\}_{n=1}^{\infty}$ rather than $\{\langle:\hat{s}:^n\rangle\}_{n=1}^{\infty}$. Consider the sequence of moments $\{\langle:\hat{s}^n:\rangle\}_{n=1}^{\infty}$ for a given quantum state $\hat{\rho}$. Formally, we can construct a characteristic function [9,10]

$$\Phi(k) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!}\langle:\hat{s}^n:\rangle = \left\langle: e^{ik\hat{s}}:\right\rangle,$$

which obeys $\Phi(0) = 1$ and $\Phi^*(-k) = \Phi(k)$. Finally, the related quasi-marginal distribution of $P(\alpha)$ can be defined as follows

$$\begin{aligned}\mu(s) &= \int \frac{dk}{2\pi}\Phi(k)e^{-iks} = \langle:\delta(s-\hat{s}):\rangle \\ &= \int d^2\alpha P(\alpha)\delta(s - s(\alpha, \alpha^*)).\end{aligned} \tag{2}$$

Therefore, if the quantum state is P-classical, then all the quasi-marginals have all the properties of bona fide probability distributions. In contrast, if any of the marginals $\mu(s)$ fail to be positive, the quantum state is P-nonclassical. The Bochner's theorem [11] provides a necessary and sufficient criterion for classical behavior of the distributions $\mu(s)$, namely that, if it holds true that

$$D_k^{(G)} = \det\left[d_{i,j}^{(G)}\right] > 0, \qquad d_{i,j}^{(G)} = \left\langle:\hat{s}^{i+j-2}:\right\rangle, \qquad 1 \leq i, j \leq k, \tag{3}$$

for all $k \in \mathbb{N}$, then $\mu(s)$ is a legitimate probability density. Note that, all moments $\left\langle:\hat{s}^n:\right\rangle$ calculated from the same quasi-marginal $\mu(s)$ corresponding to the same quasidistribution $P(\alpha)$ and in order to certify the classicality of $P(\alpha)$ one must examine the classicality of a complete class of quasi-marginals [12,13].

3. Weak measurement and its classical counterpart

Let $\hat{s}$ to be a target system observable with the corresponding eigenbasis $\{|s_k\rangle\}$. Based on the von Neumann model for a projective measurement [14], a measuring device, or pointer, with pointer position and momentum operators $\hat{X}, \hat{P}$ interacts with the system via an impulsive Hamiltonian

$$\hat{H}(t) = \varepsilon\delta(t)\hat{s}\otimes\hat{P}.$$

Just before the measurement, the pointer is in a quantum state $|\varphi_{\text{in}}\rangle$ with position uncertainty σ_X. The measurement process is then represented by the following unitary evolution

$$\hat{U} = \exp\left(-\frac{i\varepsilon}{\hbar}\hat{s}\otimes\hat{P}\right).$$

For a target prepared in the initial state $|\psi_{\text{in}}\rangle = \sum_k c_k|s_k\rangle$ the final state of the combined target-pointer is given by

$$\hat{U}\sum_k c_k|s_k\rangle|\varphi_{\text{in}}\rangle = \sum_k c_k|s_k\rangle \exp\left(-\frac{i}{\hbar}\varepsilon s_k\hat{P}\right)|\varphi_{\text{in}}\rangle.$$

The probability distribution of the pointer position right after the interaction is then given by

$$\Pr{}'(X) \equiv \Pr\left(X; |\psi_{\text{in}}\rangle, |\varphi_{\text{in}}\rangle, \{s_k\}\right) = \sum_k |c_k|^2 \left|\langle X - \varepsilon s_k|\varphi_{\text{in}}\rangle\right|^2. \tag{4}$$

Equation (4) represents a *statistical mixture* of a family of shifted forms of the initial distribution of the pointer position $\Pr(X) = |\langle X|\varphi_{\text{in}}\rangle|^2$. In projective measurements with $\sigma_X \ll \varepsilon\Delta s$ only one of the pointer positions $X_k = \varepsilon s_k$ will randomly be obtained. This leads to the spectrum $\{s_k\}$ with outcome probabilities $\{|c_k|^2\}$. In weak measurements where $\sigma_X \gg \varepsilon\Delta s$ the shifted distributions in Eq. (4) overlap, nevertheless, the probe would on average give the correct mean value $\langle\hat{s}\rangle$.

The most interesting phenomena in weak measurements is that if a projective measurement of a second observable $\hat{q}$ is carried out on the system and the eigenstate $|q_\ell\rangle$ is postselected, then the state of the probe will be given by

$$|\varphi_{\text{f}}\rangle = \sum_k \langle q_\ell|s_k\rangle c_k e^{-\frac{i}{\hbar}\varepsilon s_k\hat{P}}|\varphi_{\text{in}}\rangle.$$

Therefore, instead of the probabilistic mixture of Eq. (4), we obtain a *coherent superposition* of shifted probe states with the *new coefficients* $\langle q_\ell|s_k\rangle c_k$. In view of these considerations, probe reading $|\langle X|\varphi_{\text{f}}\rangle|^2$ conditioned by postselection in $|q_\ell\rangle$ would not satisfy usual classical results. It can be shown that, for sufficiently weak measurements, the coordinate distribution of the pointer reads as [2]

$$|\langle X|\varphi_{\text{f}}\rangle|^2 = \left|\left\langle X - \text{Re}\langle\hat{s}\rangle_{w,q}|\varphi_{\text{in}}\right\rangle\right|^2,$$

where the weak value $\langle\hat{s}\rangle_{w,q}$ is defined by

$$\langle\hat{s}\rangle_{w,q} = \frac{\langle q|\hat{\rho}\hat{s}|q\rangle}{\langle q|\hat{\rho}|q\rangle}, \qquad \hat{\rho} = |\psi_{\text{in}}\rangle\langle\psi_{\text{in}}|. \tag{5}$$

It is also possible to use the von Neumann model for classical measurement processes [2]. Suppose that both the system and the probe can be described by classical phase space distributions $W_{\text{sys}}(q, p)$ and $W_{\text{d}}(X, P)$, respectively. Further assume that the initial state of the combined system-probe is the product $W = W_{\text{sys}}(q, p)W_{\text{d}}(X, P)$. A possible model for measuring the classical quantity $s(q, p)$ consists of an impulsive interaction between the system and the probe, generating classical correlations between them, and final observation on the probe to extract distribution function $\Pr'(X)$ of the probe coordinate. Similar to the quantum scheme, the interaction Hamiltonian should be chosen as

$$H(t) = \varepsilon\delta(t)s(q, p)P.$$

The time evolution is governed by the classical Liouville's equation

$$\frac{\partial W}{\partial t} = \{H, W\} = \varepsilon\delta(t)\{s(q, p)P, W\},$$