Physics Letters A ●●● (●●●●) ●●●–●●●

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Separability criterion for quantum effects

Ikko Hamamura

Department of Nuclear Engineering, Kyoto University, 6158540 Kyoto, Japan

A R T I C L E   I N F O

Article history:
Received 27 October 2017
Received in revised form 25 June 2018
Accepted 26 June 2018
Available online xxxx
Communicated by M.G.A. Paris

Keywords:
Bell–CHSH inequality
Entanglement of observables
IBM Quantum Experience

A B S T R A C T

Entanglement of quantum states is absolutely essential for modern quantum sciences and technologies. It is natural to extend the notion of entanglement to quantum observables dual to quantum states. For quantum states, various separability criteria have been proposed to determine whether a given state is entangled. In this Letter, we propose a separability criterion for specific quantum effects (binary observables) that can be regarded as a dual version of the Bell–Clauser–Horne–Shimony–Holt (Bell–CHSH) inequality for quantum states. The violation of the dual version of the Bell–CHSH inequality is confirmed by using IBM's cloud quantum computer. As a consequence, the violation of our inequality rules out the maximal tensor product state space, that satisfies information causality and local tomography. As an application, we show that an entangled observable which violates our inequality is useful for quantum teleportation.

1. Introduction

According to the axioms of quantum theory introduced by von Neumann [1], a quantum system is associated with a separable Hilbert space and a composite quantum system is given by a tensor product of Hilbert spaces. In recent years, research to derive the axiom of Hilbert space from the physical or informational principle has gained significant momentum [2–6]. Once the axiom of the local Hilbert space is derived, observable algebra (or positive operator valued measures as observables) and Born's probability rule can immediately be derived. However, the composite system cannot be determined uniquely merely from the axiom of the local Hilbert space, so its physical justification is desired. One of the keys to characterize the composite system is the entanglement of observables because it is not compatible with the maximal tensor product state space while it satisfies information causality [7] and local tomography [8].

In this work, we develop a separability criterion for a certain class of observables. For quantum states, several separability criteria are already known, such as the positive partial transpose criterion [9,10], range criterion [11], reduction criterion [12,13], and entanglement witness [10] represented by the Bell–Clauser–Horne–Shimony–Holt (Bell–CHSH) witness [14]. Conversely, only a few attempts have been done for observables [15,16]. Our main purpose is to establish the dual version of the Bell–CHSH inequality so that one can detect entanglement of observables from violations of the inequality. Such a violation is experimentally confirmed by using IBM's cloud quantum computer. As an application, we show that observables that violate our inequality are useful for quantum teleportation [17].

The outline of this Letter is as follows: We begin by introducing the three types of positivities and fixing notations in Sec. 2. Using these positivities, we discuss the state spaces of the composite systems. In this Letter, we assume that a local quantum system is given by a finite-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^d$. In Sec. 3, we recall that some physical principles for the composite systems determine a family of possible state spaces that contains the minimal, maximal, and "physical" tensor products associated with the tensor product of Hilbert spaces $\mathbb{C}^{d_A d_B} \cong \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$. While the violation of the Bell–CHSH inequality rules out the minimal one, it does not rule out the maximal one. To overcome this problem, we propose in Sec. 4 a dual version of the Bell–CHSH inequality that can be used to exclude the maximal one. This inequality can be violated by an *entangled effect* whose definition is explained below. In Sec. 5, we show experimentally by using a quantum computer (IBM Quantum Experience) that an entangled effect violating this new inequality exists. Because the maximal composite system does not allow any entangled effect to exist, we exclude the maximal composite system from the possible candidates. In Sec. 6, we show that this violation is useful for quantum teleportation. Finally, some concluding comments are given in Sec. 7.

2. Three types of positivities

We introduce three positivities that are convenient for discussing the composite systems. An operator X on a Hilbert space $\mathcal{H}$ is *positive*, denoted by $X \geq 0$, if $\langle \psi \,|\, X\psi \rangle \geq 0$ holds for all

E-mail address: hamamura@nucleng.kyoto-u.ac.jp.

https://doi.org/10.1016/j.physleta.2018.06.043
0375-9601/© 2018 Elsevier B.V. All rights reserved.

$|\psi\rangle \in \mathcal{H}$. A bipartite operator X on a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ is *positive on pure tensors (POPT)* [18], denoted by $X \geq_{\mathrm{POPT}} 0$, if $\langle \psi \otimes \phi \,|\, X \psi \otimes \phi \rangle \geq 0$ holds for all $|\psi\rangle \in \mathcal{H}_A$ and $|\phi\rangle \in \mathcal{H}_B$. POPT is also called block positive [19,20]. A non-positive POPT operator is called an "entanglement witness" [10]. A bipartite operator X on a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ is *separable positive* and is denoted by $X \geq_{\mathrm{SP}} 0$ if X has a decomposition $X = \sum_i A_i \otimes B_i$ such that each A_i and B_i is a positive operator on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Since $\sum_i A_i \otimes B_i$ is a positive operator for positive operators A_i and B_i, $X \geq_{\mathrm{SP}} 0$ implies $X \geq 0$. Since the positive operator is positive on pure tensors, $X \geq 0$ implies $X \geq_{\mathrm{POPT}} 0$. However, the converses do not hold.

On the set of operators on a finite-dimensional Hilbert space, we introduce the Hilbert–Schmidt inner product as $\langle A \,|\, B \rangle_{\mathrm{HS}} = \mathrm{tr}\,[A^* B]$. The set of all positive operators is dual to itself via this inner product and the set of all POPT operators and the set of all separable positive operators are dual to each other.

3. No-signaling principle and local tomography

We assume that a local quantum system is associated with a finite-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^d$. If an operator ρ is positive and its trace is equal to unity, the operator ρ is called a state, which represents an experimental situation. A state space $\mathcal{S}(\mathcal{H})$ is the set of all states $\{\, \rho \in \mathcal{T}_1(\mathcal{H}) \mid \rho \geq 0 \,\}$, where $\mathcal{T}_1(\mathcal{H})$ is the set of trace class operators with unit trace. An effect space is a dual space to the state space $\mathcal{S}(\mathcal{H})$ and is defined as $\mathcal{E}(\mathcal{H}) = \{\, X \in \mathcal{B}(\mathcal{H}) \mid 0 \leq X \leq \mathbb{1} \,\}$. An effect represents a measurement apparatus that outputs "yes" or "no". Therefore an effect can be identified with a binary observable. When the state ρ is prepared, the occurrence probability of a measurement event represented by $E \in \mathcal{E}(\mathcal{H})$ is given by the trace formula $\mathrm{tr}\,[\rho E]$. This trace formula is called the generalized Born rule. From the no-signaling principle and local tomography (the latter was introduced as the global state assumption in Ref. [8]), the state space of the two quantum systems is bounded [18,8,21,22] by the minimal tensor product state space and the maximal tensor product state space, which are defined as follows: A minimal tensor product space is

$$\mathcal{S}(\mathbb{C}^{d_A}) \otimes_{\min} \mathcal{S}(\mathbb{C}^{d_B}) = \left\{ \rho \in \mathcal{T}_1(\mathbb{C}^{d_A d_B}) \;\middle|\; \rho \geq_{\mathrm{SP}} 0 \right\}, \tag{1}$$

and a maximal tensor product space is

$$\mathcal{S}(\mathbb{C}^{d_A}) \otimes_{\max} \mathcal{S}(\mathbb{C}^{d_B}) = \left\{ \rho \in \mathcal{T}_1(\mathbb{C}^{d_A d_B}) \;\middle|\; \rho \geq_{\mathrm{POPT}} 0 \right\}. \tag{2}$$

Note that the minimal tensor product space corresponds to the set of separable states. A state that is not a separable state is called an entangled state. An element of the maximal tensor product state space is called a POPT state [18,23]. The state space given by the axiom of tensor product $\mathcal{S}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$ coincides with neither of them. That is, $\mathcal{S}(\mathbb{C}^{d_A}) \otimes_{\min} \mathcal{S}(\mathbb{C}^{d_B}) \subsetneq \mathcal{S}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}) \subsetneq \mathcal{S}(\mathbb{C}^{d_A}) \otimes_{\max} \mathcal{S}(\mathbb{C}^{d_B})$ holds. Note also that this problem does not occur in classical theory. The tensor product of classical state spaces is unique [24].

The problem here is how we can distinguish $\mathcal{S}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$ experimentally from other state spaces. The left inequality is attained by the Bell's argument [25]. Let B be

$$B = \mathrm{tr}\,[\rho(A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1)], \tag{3}$$

then the Bell–CHSH inequality [14] means $|B| \leq 2$ for any separable state ρ and operators A_i and B_i satisfying $A_i^2 = B_i^2 = \mathbb{1}$ and $[A_i, B_j] = 0$. Conversely, there exists an entangled state ρ and pairs of incompatible measurements $\{\, A_0, A_1 \,\}$ and $\{\, B_0, B_1 \,\}$ that lead to the violation of the Bell–CHSH inequality [26]. The violation of the Bell–CHSH inequality thus implies the existence of an entangled state; namely, the state space of the composite system is strictly larger than $\mathcal{S}(\mathbb{C}^{d_A}) \otimes_{\min} \mathcal{S}(\mathbb{C}^{d_B})$.

4. Dual Bell–CHSH inequality

It is not a straightforward task to distinguish $\mathcal{S}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$ and $\mathcal{S}(\mathbb{C}^{d_A}) \otimes_{\max} \mathcal{S}(\mathbb{C}^{d_B})$ because there exists a quantum mechanical representation for POPT states [23]. The maximum value of the left-hand side (LHS) of the Bell–CHSH inequality in the maximal tensor product state space is $2\sqrt{2}$, which coincides with that in $\mathcal{S}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$. Furthermore, the information causality [7] is not strong enough to discard the maximal tensor product state space $\mathcal{S}(\mathbb{C}^{d_A}) \otimes_{\max} \mathcal{S}(\mathbb{C}^{d_B})$. To overcome this problem, we use the duality between a state space and an effect space. We distinguish $\mathcal{S}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$ and $\mathcal{S}(\mathbb{C}^{d_A}) \otimes_{\max} \mathcal{S}(\mathbb{C}^{d_B})$ by considering their effect spaces. The effect space dual to $\mathcal{S}(\mathbb{C}^{d_A}) \otimes_{\max} \mathcal{S}(\mathbb{C}^{d_B})$ is the set of *separable effects* $\mathcal{E}(\mathbb{C}^{d_A}) \otimes_{\min} \mathcal{E}(\mathbb{C}^{d_B}) = \left\{ X \in \mathcal{B}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}) \;\middle|\; 0 \leq_{\mathrm{SP}} X \leq_{\mathrm{SP}} \mathbb{1} \right\}$. An effect that is not a separable effect is called *an entangled effect*. We call observables that correspond to separable effect valued measures *separable observables* and observables that are not separable *entangled*. In the case of a finite outcome, these definitions are consistent with entangled measurements in Ref. [15].

Let us consider a binary measurement that has two outcomes $+1$ and -1. Let $\mathsf{M}_1, \mathsf{M}_{-1} \in \mathcal{E}(\mathbb{C}^d)$ be POVM elements with output $+1$ and -1 respectively. Let us write the Hermitian operator representing the expectation operator as $\mathsf{M} = \mathsf{M}_1 - \mathsf{M}_{-1} = 2\mathsf{M}_1 - \mathbb{1}$. We can safely identify this operator M with a binary observable $\{\, \mathsf{M}_1, \mathsf{M}_{-1} \,\}$. Let α_d be a constant $d/(d-1)$. We define *the difference from ignorance* by $E(\rho, \mathsf{M}) := \alpha_d \,\mathrm{tr}\,[(\rho - \mathbb{1}/d)\mathsf{M}]/2 = \alpha_d \,\mathrm{tr}\,[(\rho - \mathbb{1}/d)\mathsf{M}_1]$.

Lemma 1. *Let X be an operator satisfying $0 \leq X \leq \mathbb{1}$ and ρ be a state on a finite-dimensional Hilbert space $\mathbb{C}^d$. The following inequality holds:*

$$-1 \leq \alpha_d \,\mathrm{tr}\left[\left(\rho - \frac{\mathbb{1}}{d}\right) X\right] \leq 1.$$

Proof. Since ρ is self-adjoint, ρ is diagonalizable to $\rho = U \rho_D U^\dagger$ such that ρ_D is a diagonal matrix and U is a unitary matrix. Therefore,

$$\begin{aligned} &\left|\alpha_d \,\mathrm{tr}\left[\left(\rho - \frac{\mathbb{1}}{d}\right) X\right]\right| \\ &\quad = \left|\alpha_d \,\mathrm{tr}\left[\begin{pmatrix} \rho_{11} - \frac{1}{d} & & \\ & \ddots & \\ & & \rho_{dd} - \frac{1}{d} \end{pmatrix} X'\right]\right|, \end{aligned} \tag{4}$$

where $X' = U^\dagger X U$. Since X' is also an effect,

$$\begin{aligned} &= \left|\alpha_d \,\mathrm{tr}\left[\begin{pmatrix} \rho_{11} - \frac{1}{d} & & \\ & \ddots & \\ & & \rho_{dd} - \frac{1}{d} \end{pmatrix} X'\right]\right| \\ &\leq \alpha_d \sum_{\substack{i \in \{1,\ldots,d\} \\ \rho_{ii} - 1/d > 0}} \left(\rho_{ii} - \frac{1}{d}\right) \\ &\leq \alpha_d \left(1 - \frac{1}{d}\right) = 1 \end{aligned} \tag{5}$$

holds. The last inequality follows from $\sum_i \rho_{ii} = 1$ and the fact that the LHS is greater if the number of the term $-1/d$ is smaller. The last equality holds when state ρ is a pure state. □

Please cite this article in press as: I. Hamamura, Separability criterion for quantum effects, Phys. Lett. A (2018), https://doi.org/10.1016/j.physleta.2018.06.043