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On superintegrable systems separable in Cartesian coordinates

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ABSTRACT

We continue the study of superintegrable systems of Thompson's type separable in Cartesian coordinates. An additional integral of motion for these systems is the polynomial in momenta of N-th order which is a linear function of angle variables and the polynomial in action variables. Existence of such superintegrable systems is naturally related to the famous Chebyshev theorem on binomial differentials.

1. Introduction

In 1984 Thompson proved superintegrability of the Hamiltonian

$$H = p_x^2 + p_y^2 + a(x-y)^{-\frac{2}{2n-1}}, \qquad n \in \mathbb{Z}_+,$$

where n is an arbitrary positive integer [18]. To simplify the notation it is best to make a 45 degree rotation $q_1 = x + y$ and $q_2 = x - y$ as in [11]. Such superintegrable systems are still being studied up till now, see [1,9,10,16] and references within.

In this note we prove that dynamical system with Hamiltonian

$$H = p_1^2 + p_2^2 + aq_1^{M_1} + bq_2^{M_2}, \qquad a, b \in \mathbb{R}, \tag{1.1}$$

is superintegrable, if exponents M_1 and M_2 belong to the following sequence of positive rational numbers

$$M = 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \qquad n \in \mathbb{Z}_+, \tag{1.2}$$

or sequence of negative rational numbers

$$M = 0, -2, -\frac{2}{3}, -\frac{2}{5}, -\frac{2}{7}, \cdots, -\frac{2}{2n-1}. \tag{1.3}$$

These two sequences of exponents are distinguished according to the Chebyshev theorem on binomial differentials [4]. The corresponding additional first integral is a polynomial with respect to momenta p_1 and p_2.

We also discuss nonseparable systems with Hamiltonians

$$H = p_1^2 + p_2^2 + \left(aq_1^{M_1} + b\right) q_2^{M_2}, \tag{1.4}$$

where $M_{1,2}$ belong to (1.2–1.3) and present a new integrable deformation of the Fokas–Lagerstrom system [5,11]. The corresponding integral of motion is a polynomial in the momenta of the sixth degree.

2. Thompson's type systems

There are many integrable and superintegrable systems with algebraic potentials, see [3,5,8,10–12,14,15,17,18]. For arbitrary rational $M_{1,2}$ Hamiltonian H (1.1) is also an algebraic function well-defined in some part of the plane. In the same domain of definition we introduce variables

$$I_1 = p_1^2 + aq_1^{M_1}, \qquad I_2 = p_2^2 + bq_2^{M_2},$$

$$\omega_1 = -\int^{q_1} \frac{dx}{\sqrt{p_1^2 + aq_1^{M_1} - ax^{M_1}}}, \qquad \omega_2 = -\int^{q_2} \frac{dx}{\sqrt{p_2^2 + bq_2^{M_2} - bx^{M_2}}}, \tag{2.5}$$

with canonical Poisson brackets

$$\{\omega_j, I_k\} = \delta_{jk}, \qquad \{I_j, I_k\} = \{\omega_j, \omega_k\} = 0, \qquad j,k = 1,2,$$

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and equations of motion

$$\dot I_{1,2}=0,\quad \dot\omega_{1,2}=\frac{\partial H}{\partial I_{1,2}}=1,\quad \text{with}\quad H=I_1+I_2.$$

For the completely integrable system the Liouville–Arnold theorem implies that almost all points of the phase space are covered by a system of open toroidal domains with the action–angle coordinates $I_1,\dots,I_n;\omega_1,\dots,\omega_n$. In these coordinates the completely integrable system has the form

$$\dot I_k=0,\qquad \dot\omega_k=\frac{\partial H}{\partial I_k},\qquad k=1,\dots,n, \tag{2.6}$$

and symplectic structure is canonical $\Omega=\sum dI_k\wedge d\omega_k$ [2].

The variables $I_{1,2}$ and $\omega_{1,2}$ (2.5) satisfy standard equations of motion (2.6) and have canonical Poisson structure $P=\Omega^{-1}$. So, we will call them the formal action–angle variables which are well-defined functions on the original Cartesian variables only in some part of the cotangent bundle to plane.

By definition Hamiltonian H (1.1) is in the involution with action variables $I_{1,2}$ and with any function on the difference of the angle variables

$$X=F(I_1,I_2,\omega_1-\omega_2),$$

see discussion in [19–22]. Below we prove that X is the polynomial in momenta $p_{1,2}$ if $M_{1,2}$ belong to (1.2) or (1.3) because in this case $\omega_{1,2}$ are given by elementary functions. More general case when some function on difference $\omega_1-\omega_2$ are elementary functions on original variables we do not consider here, see discussion and examples in [7,19–22].

Let us recall that expressions of the form

$$x^m(\alpha+\beta x^n)^p dx,$$

where α,β are arbitrary coefficients and m,n,p are rational numbers, are called differential binomials. According to the Chebyshev theorem [4] integrals on differential binomials

$$\int x^m(\alpha+\beta x^n)^p dx,$$

can be evaluated in terms of elementary functions if and only if:

1. p is an integer, then we expand $(\alpha+\beta x^n)^p$ by the binomial formula in order to rewrite the integrand as a rational function of simple radicals $x^{j/k}$. Then we make a substitution $x=t^r$, where r is the largest of all denominators k, remove the radicals entirely and obtain integral on rational function.
2. $\dfrac{m+1}{n}$ is an integer, then we set $t=\alpha+\beta x^n$ to obtain integral
$$\int x^m(\alpha+\beta x^n)^p dx=\frac12\beta^{-\frac{m+1}{n}}\int t^p(t-\alpha)^{\frac{m+1}{n}-1}dt$$
which belongs to Case 1.
3. $\dfrac{m+1}{n}+p$ is an integer, then we transform the integral by factoring out x^n
$$\int x^m(\alpha+\beta x^n)^p dx=\int x^{m+np}(\alpha x^{-n}+\beta)^p dx.$$
The result is a new integral of the differential binomial which belongs to Case 2.

In our case (2.5) we have

$$\alpha=I_{1,2},\qquad \beta=1\qquad m=0,\qquad n=M,\qquad p=-1/2.$$

Hence action variables ω_1 and ω_2 is expressed via elementary functions only if

$$\frac1M \text{ is integer}\qquad\text{or}\qquad \frac1M-\frac12 \text{ is integer}.$$

In order to avoid logarithmic term $\ln(t)=\int t^{-1}$ in (2.5), which is also an elementary function, we have to consider only zero, positive and negative values of M, respectively.

For M_k from (1.2) action variables (2.5) are

$$M_k=0,\qquad \omega=\frac{2q_k}{p_k},$$
$$M_k=\frac{1}{n_k}>0,\qquad \omega_k=\text{polynomial of order } 2n_k-1.$$

For M_k from (1.3) action variables (2.5) are

$$M_k=0,\qquad \omega=\frac{2q_k}{p_K},$$
$$M_k=-\frac{2}{2n-1}<0,\qquad \omega=\frac{\text{polynomial of order } 2n_k-1}{I_k^{n_k}},$$

where I_k, $k=1,2$, is the corresponding action variable. Let us show a few explicit formulae for positive exponents

$$M_2=1,\qquad \omega_2=\frac{p_2}{b},$$
$$M_2=\frac13,\qquad \omega_2=\frac{p_2(3b^2q_2^{2/3}+4bq_2^{1/3}p_2^2+8/5p_2^4)}{b^3},$$

and negative exponents

$$M_2=-\frac23,\qquad \omega_2=-\frac{p_2(3bq_2^{1/3}+q_2p_2^2)}{2\left(p_2^2+bq_2^{-2/3}\right)^2},$$
$$M_2=-\frac25,\qquad \omega_2=-\frac{p_2(5bq_2^{1/5}+10/3bq_2^{3/5}p_2^2+q_2p_2^4)}{2\left(p_2^2+bq_2^{-2/5}\right)^3}.$$

Other partial or generic expressions for integrals may be found in textbooks, tables of integrals or any computer algebra system.

Proposition 1. *A Hamiltonian system defined by H (1.1) has a polynomial first integral X_N of order N, if M_1 and M_2 belong to (1.2) or (1.3):*

1. *if $M_1=1/n_1$ and $M_2=1/n_2$, then*
$$X_{2n-1}=\omega_1-\omega_2,\qquad \text{where}\qquad n=\max(n_1,n_2);$$
2. *if $M_1=-2/(2n_1-1)$ and $M_2=-2/(2n_2-1)$, then*
$$X_{2n-1}=(\omega_1-\omega_2)I_1^{n_1}I_2^{n_2},\qquad \text{where}\qquad n=n_1+n_2;$$
3. *if $M_1=1/n_1$ and $M_2=-2/(2n_2-1)$, then*
$$X_{2n-1}=(\omega_1-\omega_2)I_2^{n_2},\qquad \text{where}\qquad n=n_1+n_2;$$
4. *if $M_1=0$ and $M_2=1/n$, then*
$$X_{2n}=p_1(\omega_1-\omega_2),\qquad \text{where}\qquad p_1=\sqrt{I_1};$$
5. *if $M_1=0$ and $M_2=-2/(2n-1)$, then*
$$X_{2n}=p_1(\omega_1-\omega_2)I_2^n,\qquad \text{where}\qquad p_1=\sqrt{I_1}.$$

This integral of motion X_N is functionally independent from $I_{1,2}$ (2.5).

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