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On the nonintegrability of equations for long- and short-wave interactions

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ABSTRACT

We examine the integrability of two models used for the interaction of long and short waves in dispersive media. One is more classical but arguably cannot be derived from the underlying water wave equations, while the other one was recently derived. We use the method of Zakharov and Schulman to attempt to construct conserved quantities for these systems at different orders in the magnitude of the solutions. The coupled KdV–NLS model is shown to be nonintegrable, due to the presence of fourth-order resonances. A coupled real KdV–complex KdV system is shown to suffer the same fate, except for three special choices of the coefficients, where higher-order calculations or a different approach are necessary to conclude integrability or the absence thereof.

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1. Introduction

Systems that couple long and short waves have generated significant interest recently (*e.g.* [3,4,9,11,13]). Much attention in this area has been devoted to the following system, known as the cubic nonlinear Schrödinger–Korteweg–de Vries (NLS–KdV) system:

$$\begin{aligned} iu_t + u_{xx} + \alpha\,|u|^2\,u &= -\beta u v, \\ v_t + \gamma\, v v_x + v_{xxx} &= -\beta(|u|^2)_x, \end{aligned} \tag{1}$$

where α, β and γ are real constants, $x \in \mathbb{R}$, v is a real-valued function, and u is a complex-valued function. Recently, it was shown that (1) cannot be consistently derived starting from the underlying water wave equations [12]. The following coupled KdV–CKdV (Complex KdV) model was suggested as an alternative with a consistent derivation:

$$\begin{aligned} u_t + 2\beta u_x + \alpha u_{xxx} &= -2\beta(uv)_x, \\ v_t + \beta v_x + \beta v v_x + \gamma\, v_{xxx} &= -\beta(|u|^2)_x. \end{aligned} \tag{2}$$

As above, v (u) is a real- (complex-) valued function and α, β and γ are real constants. The coefficients in the system above occur as they are by using the scaling symmetries of the system to minimize the number of free parameters [12]. We examine whether or not the two systems (1) and (2) are integrable in a sense detailed below.

A method for showing the nonintegrability of a system developed by Zakharov and Schulman [22,23] distinguishes between *completely integrable systems* and *solvable systems*. Completely integrable systems are those for which we can find action-angle variables and solvable equations are those which can be solved by the inverse scattering transform (IST) [1]. Since integrability is a feature of the equations and not of a particular solution, we may always assume that we are working in a neighborhood of a solution with a nondegenerate linearization.

The test for complete integrability has the following steps:

1. Any completely integrable Hamiltonian system may be written locally in action-angle variables.
2. A system in action-angle variables is equivalent to a collection of uncoupled harmonic oscillators, so its Hamiltonian is quadratic.
3. Near-identity normal-form transformations [20] can be used to reduce any Hamiltonian to quadratic as long as there are no obstructions from resonances.
4. Any obstruction in the above steps due to resonances implies the system is not completely integrable.

The normal-form transformation that removes n-th order terms from the Hamiltonian gives rise to a resonance manifold which describes the process of scattering p waves ($p \in \mathbb{N}$) into $n-p$ waves. For example, if a system admits two dispersion laws $\omega^{(1)}$ and $\omega^{(2)}$, an n-th order resonance manifold is defined by

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$$M=\left\{(k_1,\dots,k_n)\in\mathbb{R}^n\,\middle|\,\sum_{j=1}^n\sigma_jk_j=0 \text{ and } \sum_{j=1}^n\sigma_j\omega^{(\ell)}(k_j)=0\right\}, \tag{3}$$

with any combination of $\sigma_j\in\{-1,1\}$ and $\ell\in\{1,2\}$. Associated with each resonance manifold is an interaction coefficient function which describes the amplitude of the scattering process. If the coefficient function vanishes on the resonance manifold then the singularity of the normal form transformation is removable and the transformation is valid. If the coefficient function does not vanish on the resonance manifold, complete integrability is not possible but solvability may be.

The test for solvability has the following steps:

1. Every system solvable by the IST has an infinite hierarchy of equations solvable by the IST. The members of the hierarchy share conserved quantities.
2. By assumption, any equation solvable by the IST is linearizable with nondegenerate linearization, so each member of the hierarchy has quadratic terms in the Hamiltonian, at least in the small amplitude limit.
3. Every member of the hierarchy has a linearly independent Hamiltonian, so the original system has infinitely many conserved quantities with linearly independent quadratic terms (see *e.g.* [16]).
4. If there exist only finitely many conserved quantities with quadratic terms for our PDE, it is not solvable by the IST.

The method of Zakharov and Schulman begins by removing all higher-order nonresonant terms as above. Next an ansatz is made about the existence of an additional conserved quantity in a power series in terms of unknown amplitudes. Upon enforcing that the quantity is independent of t, resonance manifolds appear as above. However, in this case, the resonance manifold coefficient function is multiplied by another quantity:

$$\sum_{j=1}^n\sigma_j\Phi^{(\ell)}(k_j), \tag{4}$$

where σ_j and ℓ are the same as in (3) and $\Phi^{(\ell)}$ are the unknown quadratic amplitudes in the power series. If functions $\Phi^{(\ell)}$, linearly independent from the two relations defining the resonance manifold, can be found such that (4) equals zero, then the manifold is called degenerate [18]. If any of the n-th order resonance manifolds are nondegenerate and have nonzero coefficient function, the constructed quantity is not conserved. The fact that another conserved quantity with linearly independent quadratic terms cannot be constructed implies that the system must not be solvable by the IST.

Determining whether or not a resonance manifold is degenerate poses challenges. We use the theory of web geometry [6] to check degeneracy as described in Appendix A. In Sections 2 and 3 we examine the integrability of (1) and (2).

2. Coupled NLS & KdV model

The Hamiltonian for (1) on the whole line is

$$H=\int\left(|u_x|^2+\frac{1}{2}v_x^2-\frac{\alpha}{2}|u|^4-\frac{\gamma}{6}v^3-\beta|u|^2v\right)\mathrm{d}x,$$

for the variables $z=(u,iu^*,v)$ with non-canonical Poisson structure

$$J=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&\partial_x\end{pmatrix},$$

so that (1) is equivalent to $z_t=J\delta H/\delta z$, where $\delta/\delta z$ denotes the variational gradient with respect to the components of z [5]. Here and throughout, integrals without bounds are to be interpreted as whole line integrals. This system admits two types of waves with dispersion relations $\omega_k=k^2$ and $\Omega_k=-k^3$. Here and throughout, k subscripts are indices, not partial derivatives. We introduce the Fourier transform,

$$u(x)=\frac{1}{\sqrt{2\pi}}\int u(k)e^{ikx}\mathrm{d}k=\frac{1}{\sqrt{2\pi}}\int u_ke^{ikx}\mathrm{d}k. \tag{5}$$

Applying the Fourier transform to u and v results in a Hamiltonian system for (u_k,v_k) with Hamiltonian

$$\begin{aligned}H(u_k,v_k)=&\int k^2u_ku_k^*\mathrm{d}k+\int_0^\infty k^2v_kv_k^*\mathrm{d}k\\&-\frac{\beta}{\sqrt{2\pi}}\int u_1^*v_2u_3\delta_{1-2-3}\mathrm{d}_{123}\\&-\frac{\gamma}{6\sqrt{2\pi}}\int v_1v_2v_3\delta_{123}\mathrm{d}_{123}\\&-\frac{\alpha}{2(2\pi)}\int u_1u_2u_3^*u_4^*\delta_{12-3-4}\mathrm{d}_{1234},\end{aligned} \tag{6}$$

where we use the notation $u_j=u_{k_j}$, $\mathrm{d}_{123}=\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3$, u_k^* denotes the complex conjugate of u_k, and $\delta_{12-3}=\delta(k_1+k_2-k_3)$ where $\delta(\cdot)$ is the Dirac-delta function. The integral with quadratic integrand in v_k found in (6) is reduced to an integral on the half-line using the fact that $v_k^*=v_{-k}$ since $v(x)$ is real. In Fourier variables, the dynamics are

$$i\dot{u}_k=\frac{\delta H}{\delta u_k^*},\qquad \dot{v}_k=ik\frac{\delta H}{\delta v_k^*}.$$

We introduce a_k by

$$v_k=|k|^{1/2}(a_k\theta_{-k}+a_{-k}^*\theta_k),$$

where

$$\theta_k=\theta(k)=\begin{cases}0, & k<0,\\ 1, & k\geq 0,\end{cases}$$

is the Heaviside-function. The dynamics are

$$i\dot{u}_k=\frac{\delta H}{\delta u_k^*},\qquad i\dot{a}_k=\frac{\delta H}{\delta a_k^*},$$

with

$$\begin{aligned}&H(u_k,a_k)=H_2(u_k,a_k)+H_3(u_k,a_k)+H_4(u_k,a_k),\\&H_2(u_k,a_k)=\int\omega_ku_ku_k^*\mathrm{d}k+\int_{-\infty}^0\Omega_ka_ka_k^*\mathrm{d}k,\\&H_3(u_k,a_k)=\int U_{123}(a_1^*a_2a_3+a_1a_2^*a_3^*)\delta_{1-2-3}\mathrm{d}_{123}\\&\qquad+\int V_{123}(u_1^*a_2u_3+u_1a_2^*u_3^*)\delta_{1-2-3}\mathrm{d}_{123},\\&H_4(u_k,a_k)=\int W_{1234}u_1u_2u_3^*u_4^*\delta_{12-3-4}\mathrm{d}_{1234},\\&U_{123}=-\frac{\gamma}{2\sqrt{2\pi}}|k_1k_2k_3|^{1/2}\theta_{-1}\theta_{-2}\theta_{-3},\\&V_{123}=-\frac{\beta}{\sqrt{2\pi}}|k_2|^{1/2}\theta_{-2},\\&W_{1234}=-\frac{\alpha}{2(2\pi)}.\end{aligned} \tag{7}$$

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