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Deterministic quantum dense coding networks

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ABSTRACT

We consider the scenario of deterministic classical information transmission between multiple senders and a single receiver, when they a priori share a multipartite quantum state – an attempt towards building a deterministic dense coding network. Specifically, we prove that in the case of two or three senders and a single receiver, generalized Greenberger–Horne–Zeilinger (gGHZ) states are not beneficial for sending classical information deterministically beyond the classical limit, except when the shared state is the GHZ state itself. On the other hand, three- and four-qubit generalized W (gW) states with specific parameters as well as the four-qubit Dicke states can provide a quantum advantage of sending the information in deterministic dense coding. Interestingly however, numerical simulations in the three-qubit scenario reveal that the percentage of states from the GHZ-class that are deterministic dense codeable is higher than that of states from the W-class.

1. Introduction

The rapid development of quantum information science is largely due to discoveries of communication protocols [1–6] by using entangled quantum states [7]. Their successful realizations in physical systems like photons [8], ions [9], superconducting qubits [10], nuclear magnetic resonance (NMR) [11] etc. also make the field attractive. When a priori an entangled state is shared between sender(s) and receiver(s), tasks of communication protocols can broadly be classified in two categories – classical information [3,1,2] and quantum state transfer [4]. The former, without the security issue during the transmission of information, is known as the quantum dense coding protocol (DC) [3] which is the main theme of this rapid communication. The quantum DC protocol has been experimentally implemented with photons [12] and later with NMR [13], trapped ions [14], and also in continuous variable systems [15].

The original DC protocol describes the advantage to send the information of N possible outcomes of a classical random variable, say X, when encoded in a quantum state from a single sender (Alice) to a single receiver (Bob). Bennett and Wiesner [3] have shown that if Alice and Bob share the maximally entangled singlet state, $|\psi^-_{AB}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, Alice can transform the state into four possible orthogonal states by acting local unitaries on her part and can send $\log_2 4 = 2$ bits of classical information by sending only a single spin-1/2 particle, i.e., a two-dimensional system. If the initially shared state is $|\phi_{AB}\rangle = a|01\rangle + b|10\rangle$, where $a, b \in \mathbb{C}$ with $\mathbb{C}$ being the set of complex numbers and $|a|^2 + |b|^2 = 1$ with $a \neq b$ (both non-zero), or an arbitrary state, ρ_{AB}, Alice can no longer create orthogonal output states by performing unitaries and hence the receiver gets less information. In the asymptotic limit, when many copies of ρ_{AB} are provided, the amount of maximal classical information transferred on an average is the dense coding capacity (C) [16,17], given by $C(\rho_{AB}) = \log_2 d_A + \max\{S(\rho_B) - S(\rho_{AB}), 0\}$, where d_A is the dimension of the Hilbert space of the sender's subsystem, $S(\sigma) = -\mathrm{tr}(\sigma \log_2 \sigma)$ is the von Neumann entropy of σ, and $\rho_B = \mathrm{tr}_A(\rho_{AB})$ is the reduced density matrix of the receiver's subsystem. The first term is the classical limit for information transfer, while the remaining terms quantify the quantum advantage in DC. Clearly, in the case of pure states, the entanglement content of the shared state [18] and the quantum advantage of DC capacity is equal.

Instead of considering an asymptotic way of transferring classical bits which also is probabilistic in nature, we deal with a DC scheme in a single-copy level, using a shared non-maximally entangled pure state, where Alice encodes the information by performing unitary operations on her part in such a way that upon receiving the entire system, Bob can always distinguish the output states without any error, i.e., deterministically, by performing global measurements. Such protocol for a single sender and a single receiver was introduced in Ref. [19], and referred to as the deterministic dense coding (DDC) protocol [20]. Since the protocol is at the single-copy level, it is also important from an experimental point of view [21]. In DDC, Alice's aim is to find unitary

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operators, $\{U_i^A\}$, such that mutually orthogonal states can be created by applying $\{U_i^A\}$ on her part of $|\psi_{AB}\rangle$, thereby distinguishing them by Bob using global measurements. If $|\psi_{AB}\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$, where d is the dimension of each subsystem, the classical limit of the alphabet-size of the message is d, while $|\psi_{AB}\rangle$ is said to be deterministically dense codeable if the maximal number of orthogonal unitary operators, N^{ψ}_{max}, is greater than d. It was proven that the entire family of pure states in $\mathbb{C}^2 \otimes \mathbb{C}^2$ except the maximally entangled state [19] is useless for deterministic dense coding. Till now, all the studies on DDC are restricted to a single sender and a single receiver (cf. [22]), although the importance of building a communication protocol between several senders and several receivers is unquestionable. In this work, we address the question of building a DDC network between several senders and a single receiver. Interestingly, we show that DDC is possible even with two-level systems already if one increases the number of senders to two. We first prove that the DDC protocol with quantum advantage is *not* possible when the shared state is a generalized GHZ state with two or more than two senders and a single receiver except when it is a GHZ state for which DDC and DC attain the maximum capacities. We show that the DDC scheme can be executed by using the generalized W states beyond the classical limit. We also perform a comparison between the states from the GHZ- and the W-classes according to their usefulness in DDC. Moreover, we comment that the maximal number of unitaries cannot reach $d^{M+1} - 1$, when a $(M+1)$-party state is shared between M senders and a single receiver, each having dimension d (cf. [23] for two-qubit states).

2. Deterministic dense coding network: many senders and a single receiver

We now extend the deterministic dense coding protocol to multiple senders and a single receiver, situated in distant locations. Let us consider a $(M+1)$-party pure state $|\psi_{S_1 S_2 \ldots S_M R}\rangle$ shared between the M senders, $S_1, S_2, \ldots, S_M$, and a single receiver, R. A set of arbitrary local unitary operators, $\{U_i^{S_k}\}$ is performed by each sender, S_k. Our task is to find out the maximal number of unitaries of the form $\{\bigotimes_k U_i^{S_k}\}$ such that the members of the set of output states $\{\bigotimes_k U_i^{S_k} \otimes \mathbb{I}^R |\psi_{S_1 S_2 \ldots S_M R}\rangle\}$, sent to the receiver, are mutually orthogonal to each other. Hence, we find $\{U_i^{S_k}\}$, satisfying

$$\langle \psi_{S_1 S_2 \ldots S_M R}| \left(\bigotimes_k U_i^{S_k \dagger} \otimes \mathbb{I}^R \right) \left(\bigotimes_{k'} U_j^{S_{k'}} \otimes \mathbb{I}^R \right) |\psi_{S_1 S_2 \ldots S_M R}\rangle = \delta_{ij}, \quad (1)$$

or, alternatively

$$\mathrm{tr}\left((\bigotimes_k U_i^{S_k \dagger}) \rho_{S_1 S_2 \ldots S_M} (\bigotimes_{k'} U_j^{S_{k'}}) \right) = \delta_{ij}, \quad (2)$$

where $\rho_{S_1 S_2 \ldots S_M} = \mathrm{tr}_R \left(|\psi_{S_1 S_2 \ldots S_M R}\rangle \langle \psi_{S_1 S_2 \ldots S_M R}| \right)$ is the reduced density matrix of all the senders' subsystems for a given state $|\psi_{S_1 S_2 \ldots S_M R}\rangle$. The aim is to find the maximal number of such unitaries, N^{ψ}_{max}, which will define the alphabet-size of the message that senders can send. We can always choose the identity operator $\bigotimes_k \mathbb{I}_{S_k}$ on the Hilbert space of the senders as one of the members of the above set of orthogonal unitary matrices $\{\bigotimes_k U_i^{S_k}\}$. The task then reduces to find remaining $N^{\psi}_{max} - 1$ number of unitary matrices, satisfying Eq. (2), either analytically or by numerical simulations. It is noteworthy to mention that, in general, N^{ψ}_{max} lies in the range $[d^M, d^{M+1}]$, where d^M is the classical limit and d^{M+1} is the quantum limit of the alphabet-size. For a given state, $|\psi\rangle$, if we find that $N^{\psi}_{max} > d^M$, we conclude that the state has quantum advantage in DDC.

Let us restrict ourselves to two senders and a single receiver. They now share a three-qubit pure state $|\psi_{S_1 S_2 R}\rangle$ and each sender performs a two-dimensional unitary operator given by

$$U_i^{S_k} = \begin{pmatrix} \cos\theta_i^{S_k} e^{i x_i^{S_k}} & -\sin\theta_i^{S_k} e^{i y_i^{S_k}} \\ \sin\theta_i^{S_k} e^{-i y_i^{S_k}} & \cos\theta_i^{S_k} e^{-i x_i^{S_k}} \end{pmatrix}, \quad (3)$$

where $\theta_i^{S_k} \in [0, \pi]$ and $x_i^{S_k}, y_i^{S_k} \in [0, 2\pi]$. Notice that we have chosen $U_i^{S_k}$ as an element of $SU(2)$, since any arbitrary value of the determinant does not contribute to the orthogonality condition except a global phase. In this case, Eq. (2) reduces to

$$\mathrm{tr}\big((U_i^{S_1 \dagger} \otimes U_i^{S_2 \dagger}) \rho_{S_1 S_2} (U_j^{S_1} \otimes U_j^{S_2}) \big) = \delta_{ij}. \quad (4)$$

We will show that unlike two-qubit states, for three-qubit pure states, the solution of Eq. (4) exists, thereby ensuring quantum advantage by DDC scheme. A similar observation can also be made for a higher number of senders.

3. DDC: GHZ-class vs. W-class

Let us first consider two important families of three-qubit states. They are the generalized GHZ (gGHZ) states [24], given by

$$|gGHZ_{S_1 S_2 R}\rangle = \sqrt{\alpha}|000\rangle + \sqrt{1-\alpha}\, e^{i\mu}|111\rangle, \quad (5)$$

where $\alpha \in [0, 1]$ and $\mu \in [0, 2\pi)$, and the generalized W (gW) states [25],

$$|gW_{S_1 S_2 R}\rangle = \sqrt{\alpha}|001\rangle + \sqrt{\beta}|010\rangle + \sqrt{1-\alpha-\beta}|100\rangle \quad (6)$$

with $\alpha, \beta \in [0, 1]$, and $\alpha + \beta \leq 1$. For $\alpha = \frac{1}{2}$ in Eq. (5), we get the well known GHZ state, while in Eq. (6) we have the W state for $\alpha = \beta = \frac{1}{3}$. The gGHZ states and the gW states are well known subsets (of measure zero) of two SLOCC (stochastic local operations and classical communication) inequivalent classes of three-qubit pure states [26], namely, the GHZ-class [27] and the W-class [28] respectively. The set of tripartite states, that can be converted into the GHZ state using only SLOCC, defines the GHZ-class, whereas the W-class contains all the tripartite states that can be converted, by means of SLOCC, into the W state. These two classes are inequivalent in the sense that one cannot convert, with finite probability, a member of the GHZ-class into a member of the W-class, or vice-versa, using SLOCC. We will prove that although the gGHZ states (subset of GHZ-class) is not good for DDC, the quantum advantage of DDC is possible using the gW states (subset of W-class) by showing N_{max} beyond the classical limit.

3.1. No DDC for generalized Greenberger–Horne–Zeilinger states

Suppose the shared state is unentangled in sender and receiver bipartition, then the maximum amount of information that the two senders, each having two-dimensional systems can send to the receiver is two bits. Moreover, the capacity of dense coding with the GHZ state reaches its maximum value, implying successful implementation of DDC protocol with $N^{GHZ}_{max} = 8$ [17]. Let us consider DDC by using gGHZ states with $\alpha \neq \frac{1}{2}$. For several reasons, including that in Theorem 1 below, the GHZ state is considered to be the "maximally entangled" among gGHZ states. See Ref. [29] in these regards. We therefore refer to gGHZ states with $\alpha \neq \frac{1}{2}$ as "non-maximally entangled" gGHZ states.

Theorem 1. *Non-maximally entangled generalized Greenberger–Horne–Zeilinger states are not useful for deterministic dense coding with two senders and a single receiver.*