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Circuit bounds on stochastic transport in the Lorenz equations

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ABSTRACT

In turbulent Rayleigh–Bénard convection one seeks the relationship between the heat transport, captured by the Nusselt number, and the temperature drop across the convecting layer, captured by the Rayleigh number. In experiments, one measures the Nusselt number for a given Rayleigh number, and the question of how close that value is to the maximal transport is a key prediction of variational fluid mechanics in the form of an upper bound. The Lorenz equations have traditionally been studied as a simplified model of turbulent Rayleigh–Bénard convection, and hence it is natural to investigate their upper bounds, which has previously been done numerically and analytically, but they are not as easily accessible in an experimental context. Here we describe a specially built circuit that is the experimental analogue of the Lorenz equations and compare its output to the recently determined upper bounds of the stochastic Lorenz equations [1]. The circuit is substantially more efficient than computational solutions, and hence we can more easily examine the system. Because of offsets that appear naturally in the circuit, we are motivated to study unique bifurcation phenomena that arise as a result. Namely, for a given Rayleigh number, we find a reentrant behavior of the transport on noise amplitude and this varies with Rayleigh number passing from the homoclinic to the Hopf bifurcation.

1. Introduction

The Lorenz equations are an archetype for key aspects of nonlinear dynamics, chaos and a range of other phenomena that manifest themselves across all fields of science, particularly in fluid flow [see e.g., 2,3]. Lorenz [4] derived his model to describe a simplified version of Saltzman's treatment of finite amplitude convection in the atmosphere [5]. The three coupled Lorenz equations, which initiated the modern field we now call chaos theory, are

$$\begin{aligned} \dot{x} &= \sigma(y-x), \\ \dot{y} &= \rho x - xz - y \quad \text{and} \\ \dot{z} &= xy - \beta z, \end{aligned} \tag{1}$$

where x describes the intensity of convective motion, y the temperature difference between ascending and descending fluid and z the deviation from linearity of the vertical temperature profile. The parameters are the Prandtl number σ, the normalized Rayleigh number, $\rho = \frac{\mathrm{Ra}}{\mathrm{Ra}_c}$, where $\mathrm{Ra}_c = \frac{27\pi^4}{4}$, and a geometric factor β. Here we take $\sigma = 10$ and $\beta = \frac{8}{3}$, the original values used by Lorenz.

The sensitivity of solutions to small perturbations in initial conditions and/or parameter values characterize chaotic dynamics and have a wide array of implications. Chaotic behavior does not lend itself well to standard analysis, but modern computational methods provide us with vastly more powerful tools than those available to Lorenz. However, one powerful mathematical method used for example in the study of fluid flows is variational, and assesses the optimal value of a transport quantity, or a bound [6–8], which we briefly discuss next.

1.1. Bounds on fluid flows

Bounding quantities in fluid flows has important physical consequences and substantial theoretical significance. Whereas variational principles are central when an action is well-defined and phase space volume is conserved, they pose significant challenges for dissipative nonlinear systems in which the phase space volume is not conserved and thus not Hamiltonian [e.g., 9]. However, initiated by the work of Howard [10], who used a variational approach to determine the upper bounds on heat transport in statistically stationary Rayleigh–Bénard convection, with incompressibility as one of the constraints, the concept of mathematically bounding

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the behavior of a host of flow configurations has developed substantially [8], as well as in other dissipative systems such as solidification [11].

Transport in the Lorenz system is defined as the quantity $\langle xy \rangle$ where $\langle \cdot \rangle$ denotes the infinite time average. We also note that this quantity is proportional to $\langle z \rangle$ and $\langle x^2 \rangle$. Bounds on transport were first produced by Malkus [12] and Knobloch [13] in the 1970s. Knobloch used the theory of stochastic differential equations to analyze statistical behavior in the Lorenz system, with particular focus on the computation of long time averages, including the transport. His method can be seen as an early incarnation of the background method of Constantin and Doering [7]. Following the development of new analytical tools, interest in bounds on the Lorenz system and their interpretation has grown over the past two decades. Using the background method, Souza and Doering [14] produced sharp upper bounds on the transport $\langle xy \rangle \le \beta(\rho-1)$, which are saturated by the non-trivial equilibrium solutions $(x, y, z) \equiv (x_0, y_0, z_0) = (\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho-1)$. Agarwal and Wettlaufer [1] extended their result to the stochastic Lorenz system, recovering the sharp bounds in the zero noise amplitude limit.

Recent numerical work, especially in the form of semi-definite programming, has provided novel methods for bounding and locating optimal trajectories, that is, trajectories that maximize some function of a system's state variables [15].

Tobasco et al. [16] describe such an approach to this problem through the use of auxiliary functions similar to Lyapunov functions used in stability analyses. Goluskin [17] utilizes this method to compute example bounds on polynomials in the Lorenz system. For transport in particular (the polynomial xy), his results agree with the existing analytical theory. In the chaotic regime, however, we know the optimal solutions are unstable and are only attained for a very specific set of initial conditions, and in the stochastic system such solutions may never be realized. Hence, it is natural to ask about bounds on non-specious trajectories. Fantuzzi et al. [18] present a semi-definite programming approach to this problem similar to that of Tobasco et al. [16], though work still needs to be done to apply their methods to systems containing unstable limit cycles and saddle point equilibria, which includes the Lorenz system.

We offer an alternative method for analyzing time-averaged behavior through the use of an analog circuit. Circuits can model a wide range of linear and nonlinear dynamical systems, and by collecting voltage data from the circuit we can perform calculations of any function of the systems state variables. In this paper, we use the circuit approach to study transport, $\langle xy \rangle$, in the stochastic Lorenz system, a choice which is motivated by its physical analogy with Rayleigh–Bénard convection. For true convective motion, experimental measurements of transport are challenging, and the circuit provides us with a quick and easy way to perform these calculations, in fact much faster than standard numerical methods. We first introduce the stochastic Lorenz system and the corresponding bounds on transport. We then discuss the circuit implementation and offer an analytical model for the circuit system. Finally, we discuss our computations of transport in relation to the analytical upper bound theory and compare our results to the numerical solutions.

2. The circuit Lorenz experiment

2.1. Upper bounds of the stochastic Lorenz system

The Lorenz system might be best described as a *motif* of atmospheric convection, which was the motivation for its derivation. However, such physically based models can often become more realistic by adding a stochastic element to account for random fluctuations, observational error, and unresolved processes. This conceptually common idea has become particularly popular in climate modeling and weather prediction [e.g., 19,20]. Here, we follow this approach in the Lorenz system by adding a stochastic term with a constant coefficient [1] viz.,

$$\begin{aligned}\dot{x} &= \sigma(y-x) + A\xi_x,\\ \dot{y} &= \rho x - xz - y + A\xi_y \quad \text{and}\\ \dot{z} &= xy - \beta z + A\xi_z,\end{aligned} \tag{2}$$

where the ξ_i are Gaussian white noise processes, A is the noise amplitude and σ, β, and ρ are as in Equations (1). The circuit described below in §2.2 allows us to experimentally test and analyze stochastic bounds of the transport in the Lorenz system subject to forced and intrinsic noise. In the infinite time limit, the stochastic upper bounds of Agarwal and Wettlaufer [1] are given by

$$\langle xy \rangle_T \le \beta(\rho-1) + \frac{A^2}{\rho-1}\Big(1 + \frac{1}{2\sigma}\Big). \tag{3}$$

For $A = 0$ these reduce to the upper bounds of Souza and Doering [14]. However, unlike the deterministic case, the fixed point solutions do not exist so that the optimum is never truly attained. We note that these bounds tend to infinity as $\rho \to 1$, though Fantuzzi [21] improved this bound in the low Rayleigh number regime.

2.2. The Lorenz electrical circuit

Following the implementation described by Horowitz [22], the Lorenz system is modeled in an analog circuit through a series of op-amp integrators and voltage multipliers (Fig. 1). Mathematically, this implementation essentially solves Equations (1) by continuously integrating both sides and returning the output x, y, z back into the circuit. Adding a noise element to the integrators allows us to adapt this circuit to the stochastic Lorenz systems. To generate noise we use Teensy 3.5 microprocessors. These boards possess hardware random number generators that provide a higher quality of randomness compared to those more commonly found on microprocessors and computers. They have 12-bit resolution digital to analog converters (DAC) allowing us to output a voltage between 0 V and 3.3 V at $2^{12} = 4096$ discrete values. This gives us better spectral characteristics compared to pulse-width modulation which outputs either 0 V or 3.3 V with a duty cycle that corresponds to the analog level. To achieve Gaussian random noise we sum 8 random integers, chosen in a limited range corresponding to the noise amplitude. The number is then centered about the middle voltage corresponding to the integer 2048 and outputed through the DAC channel. Following this process the signal is AC coupled to ensure the voltage is symmetric about 0 V, and further amplification is achieved through an op-amp. This method allows us to easily control the noise processes and amplitudes directly from the computer, and thus to automate many components of the experiment.

To collect voltage data from the circuit, we use an Arduino Due microprocessor with 12-bit analog read resolution along with several voltage dividers and amplifiers to put the voltages in the Arduino's operating range of 0–3.3 V. As in the case of the noise generation, when processed the voltages appear as an integer between 0 and 4095, corresponding to a voltage between 0–3.3 V. From this data we can convert back to the original voltage using measurements of the amplifiers and voltage dividers and scaling by 10, the normalization factor of the circuit.

The rate of integration is determined by the three capacitors, ideally equal in value. This allows us to adjust the sampling rate depending on the application. For measuring transport, we can