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Radial position-momentum uncertainties for the infinite circular well and Fisher entropy

Ariadna J. Torres-Arenas [a], Qian Dong [a], Guo-Hua Sun [b], Shi-Hai Dong [a,*]

[a] Laboratorio de Información Cuántica, CIDETEC, Instituto Politécnico Nacional, UPALM, CDMX 07700, Mexico
[b] Catedrática CONACyT, Centro de Investigación en Computación, Instituto Politécnico Nacional, UPALM, CDMX 07738, Mexico

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ABSTRACT

We show how the product of the radial position and momentum uncertainties can be obtained analytically for the infinite circular well potential. Some interesting features are found. First, the uncertainty Δr increases with the radius R and the quantum number n, the n-th root of the Bessel function. The variation of the Δr is almost independent of the quantum number n for $n > 4$ and it will arrive to a constant for a large n, say $n > 4$. Second, we find that the relative dispersion $\Delta r/\langle r\rangle$ is independent of the radius R. Moreover, the relative dispersion increases with the quantum number n but decreases with the azimuthal quantum number m. Third, the momentum uncertainty Δp decreases with the radius R and increases with the quantum numbers $m > 1$ and n. Fourth, the product $\Delta r\Delta p_r$ of the position-momentum uncertainty relations is independent of the radius R and increases with the quantum numbers m and n. Finally, we present the analytical expression for the Fisher entropy. Notice that the Fisher entropy decreases with the radius R and it increases with the quantum numbers $m > 0$ and n. Also, we find that the Cramer–Rao uncertainty relation is satisfied and it increases with the quantum numbers $m > 0$ and n, too.

1. Introduction

Since the introduction of the famous uncertainty principle, its frequently quoted statement says: "It is impossible to know both the position and the momentum of a particle at a given momentum to an arbitrary degree of accuracy" [1], which is expressed as $\Delta x\Delta p \geq \hbar/2$.

It is usually known that the position-momentum uncertainty products for the harmonic oscillator and non-relativistic hydrogen atom were obtained analytically 87 years ago [2]. After that the position-momentum uncertainty relations for a few soluble potentials were also worked out [3]. Note that almost all contributions to this topic are limited to one-dimensional Schrödinger equation. Except for these studies, Bialynicki–Birula et al. proposed so-called BBM inequality which generalized the quantum mechanical Heisenberg-like uncertainty relation for 3-dimensional one-electron systems [4]. Recently, the radial position-momentum uncertainties for the non-relativistic hydrogen-like atoms in three dimensions were studied [5]. On the other hand, we have also studied the uncertainties of the Dirac and Klein–Gordon equations for the hydrogen-like atoms [6,7]. Motivated by these contributions, we are going to study the uncertainty relations for the infinite circular well. Undoubtedly, such a study will open a new research direction in the field of the confined systems.

This paper is organized as follows. In Section 2 we first briefly review the exact solutions of the infinite circular well and then we present the uncertainty relations Δr, Δp_r and $\Delta r\Delta p_r$ for this system. In Section 3 we study the Fisher entropy and derive the Cramer–Rao uncertainty relation. Some concluding remarks are given in Section 4.

* Corresponding author.
E-mail addresses: sunghdb@yahoo.com (G.-H. Sun), dongsh2@yahoo.com (S.-H. Dong).

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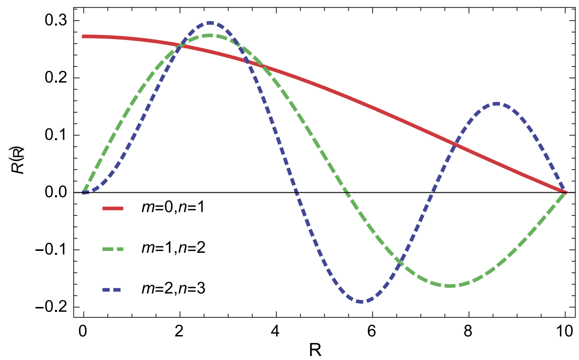

Fig. 1. Radial wave functions for the states $m=0,n=1$, $m=1,n=2$ and $m=2,n=3$, respectively.

2. Analytical results for the uncertainty relations Δr, Δp_r and $\Delta r\Delta p_r$

The Schrödinger equation for an infinite circular well defined as $V(r)=0$ $(r\le R)$ and $V(r)=\infty$ $(r>R)$ in relevant polar coordinates has been studied well in almost all textbooks and also in our recent work [8]. For simplicity, we only write out the radial wave functions and eigenvalues as [8]

$$R_{mn}(r)=N_m J_m(k_{mn}r)=N_{mn}J_m(\alpha_{mn}r/R),$$
$$E_{mn}=\frac{\hbar^2k_{mn}^2}{2\mu}=\frac{\hbar^2}{2\mu R^2}\alpha_{mn}^2,\quad N_{mn}=\frac{\sqrt{2}}{R|J_{m+1}(\alpha_{mn})|},\tag{1}$$

where the α_{mn} is the n-th root of the Bessel function $J_m(x)$. All wave functions are equal to zero for $r=R$. We illustrate the radial wave functions for $m=0,n=1$, $m=1,n=2$ and $m=2,n=3$ in Fig. 1.

We now go about to study the uncertainty relations for the infinite circular well. In order to calculate the uncertainty Δr, we have to know the mean values of r and r^2 for an electron moving inside the circular well. Before studying them, let us consider a more general case r^s. After some algebraic manipulations, we have

$$\langle r^s\rangle=\frac{2^{1-2m}R^s(\alpha_{mn})^{2m}\,{}_2F_3\left(m+\frac{1}{2},m+\frac{s}{2}+1;m+1,2m+1,m+\frac{s}{2}+2;-(\alpha_{mn})^2\right)}{(2m+s+2)\Gamma(m+1)^2\,|J_{m+1}(\alpha_{mn})|^2}.\tag{2}$$

For the special cases $s=1,2$, which are related to the present uncertainty relations, we have the following results

$$\langle r\rangle=\frac{2^{1-2m}R(\alpha_{mn})^{2m}\,{}_2F_3\left(m+\frac{1}{2},m+\frac{3}{2};m+1,m+\frac{5}{2},2m+1;-(\alpha_{mn})^2\right)}{(2m+3)\Gamma(m+1)^2\,|J_{m+1}(\alpha_{mn})|^2},\tag{3}$$

and

$$\langle r^2\rangle=\frac{2^{-2m}(m+1)R^2(\alpha_{mn})^{2m}\,{}_2F_3\left(m+\frac{1}{2},m+2;m+1,m+3,2m+1;-(\alpha_{mn})^2\right)}{\Gamma(m+1)\Gamma(m+3)\,|J_{m+1}(\alpha_{mn})|^2}.\tag{4}$$

The uncertainty Δr in measuring the distance of the electron from the nucleus is calculated as follows:

$$\begin{aligned}\Delta r&=\sqrt{\langle r^2\rangle-\langle r\rangle^2}\\&=\frac{2^{-2m}R(\alpha_{mn})^m}{|J_{m+1}(\alpha_{mn})|^2}\Bigg\{\Bigg[4^m(m+1)\Gamma(2m+1)\,|J_{m+1}(\alpha_{mn})|^2\,{}_2\tilde{F}_3\left(m+\frac{1}{2},m+2;m+1,m+3,2m+1;-(\alpha_{mn})^2\right)\\&\quad-\frac{4(\alpha_{mn})^{2m}\,{}_2F_3\left(m+\frac{1}{2},m+\frac{3}{2};m+1,m+\frac{5}{2},2m+1;-(\alpha_{mn})^2\right)^2}{(2m+3)^2\Gamma(m+1)^4}\Bigg]\Bigg\}^{1/2},\end{aligned}\tag{5}$$

which means that Δr is proportional to the radius R as shown in Fig. 2. How to understand this? We are able to explain it by considering the fact that the electron can move more freely when the radius R of the circular well becomes large. Moreover, the Δr increases with the quantum number n, but decreases with the m (see Fig. 3). On the other hand, the relative dispersion $\Delta r/\langle r\rangle$ in the measurement of radial position, which also depends on the quantum numbers m and n, but independent of the radius R (see Fig. 4), will give us a possible estimate of how indeterminate the measurement in radial position r is relative to the mean value $\langle r\rangle$ for an electron confined in the infinite circular well. We find that the relative dispersion $\Delta r/\langle r\rangle$ increases with the quantum number n but decreases with the quantum number m and will arrive to zero for a larger m. This means that the electron is more stable when the m is very large. When the quantum number n becomes large, the $\Delta r/\langle r\rangle$ increases for a given m and finally arrives to a constant when the n becomes larger as displayed in Fig. 5.

In order to obtain the product $\Delta r\Delta p_r$ of the uncertainties Δr and Δp_r, we have to study the radial momentum uncertainty Δp_r. For this purpose, we begin by considering the radial momentum operator in two dimensions

$$p_r=-i\hbar\left(\frac{\partial}{\partial r}+\frac{1}{2r}\right),\tag{6}$$