Physics Letters A ••• (••••) •••–•••

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Spectral density of mixtures of random density matrices for qubits

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ARTICLE INFO

Article history:
Received 3 March 2018
Received in revised form 15 March 2018
Accepted 6 April 2018
Available online xxxx
Communicated by M.G.A. Paris

Keywords:
Horn's problem
Probability density function
Random quantum state
Probabilistic mixture

ABSTRACT

We derive the spectral density of the equiprobable mixture of two random density matrices of a two-level quantum system. We also work out the spectral density of mixture under the so-called quantum addition rule. We use the spectral densities to calculate the average entropy of mixtures of random density matrices, and show that the average entropy of the arithmetic-mean-state of n qubit density matrices randomly chosen from the Hilbert–Schmidt ensemble is never decreasing with the number n. We also get the exact value of the average squared fidelity. Some conjectures and open problems related to von Neumann entropy are also proposed.

1. Introduction

In the early 1950s, physicists had reached the limits of deterministic analytical techniques for studying the energy spectra of heavy atoms undergoing slow nuclear reactions. It is well-known that a random matrix with appropriate symmetries might serve as a suitable model for the Hamiltonian of the quantum mechanical system that describes the reaction [1]. The eigenvalues of this random matrix model the possible energy levels of the system [2]. In quantum statistical mechanics, the canonical states of the system under consideration are the reduced density matrices of the uniform states on a subspace of system and environment. Moreover, such reduced density matrices can be realized by Wishart matrix ensemble [3]. Thus investigations by using random matrix theoretical techniques can lead to deeper insightful perspectives on some problems in Quantum Information Theory [4–10]. In fact, most works using RMT as a tool to study quantum information theory are concentrated on the limiting density and their asymptotics. In stark contrast, researchers obtained an exact probability distribution of eigenvalues of a multipartite random quantum state via deep mathematical tools such as symplectic geometric method albeit the used definition of Duistermaat–Heckman measure is very abstract and difficult [11,12]. Besides, the authors conducted exact and asymptotic spectral analysis of the difference between two random mixed quantum states [13]. Non-asymptotic results about average quantum coherence for a random quantum state [14–16] and its typicality were obtained recently. Motivated by the connection of the works [11,12] and Horn's problem [17], we focus the spectral analysis of mixture of several random states in a two-level system. Although the spectral analysis of superposition of random pure states were performed recently [18,19], the topic about the spectral densities for mixtures of random density matrices from two quantum state ensembles is rarely discussed previously.

Along this line, we will make an attempt toward exact spectral analysis of two kinds of mixtures of two random density matrices for qubits: a) equiprobable mixture of two random density matrices, based on the results obtained in Ref. [17], and b) mixture of two random density matrices under the quantum addition rule (see Definition 3.4, [20]). To the best of our knowledge, such kind of spectral analysis for mixture of random states is rarely conducted, in particular the spectral density under the quantum addition rule. The aim of this work is to analyze properties of a generic quantum state on two-dimensional Hilbert space. For two random states chosen from two unitary orbits, each distributed according to Haar measure over $\mathrm{SU}(2)$, we derive the spectral density of the equiprobable mixture of both random density matrices for qubits, and the spectral density of mixture of both random density matrices under the quantum addition rule. When they are distributed according to the Hilbert–Schmidt measure in the set $\mathrm{D}(\mathbb{C}^2)$, i.e., the set of all 2×2 density matrices, of quantum states of dimension two, we can calculate the average entropy of ensemble generated by two kinds of mixtures. We also study entropy inequality under the quantum addition rule.

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https://doi.org/10.1016/j.physleta.2018.04.018
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The paper is organized as follows: In Section 2, we recall some useful facts about a qubit. Then we present our main results with their proofs in Section 3. Specifically, we obtain the spectral densities of two kinds of mixtures of two qubit density matrices: (a) the equiprobable mixture and (b) the mixture under the quantum addition rule. By using the relationship between an eigenvalue of a qubit density matrix and the length of its Bloch vector representation, we get compact forms (Theorem 3.2 and Theorem 3.7) of corresponding spectral densities. We also investigate a quantum Jensen–Shannon divergence-like quantity, based on the mixture of two random density matrices under the quantum addition rule. It provide a universal lower bound for the quantum Jensen–Shannon divergence. However our numerical experiments show that such lower bound cannot define a true metric. Next, in Section 4, we use the obtained results in the last section to calculate the average entropies of mixtures of two random density matrices in a two-level quantum system. We show that the average entropy of the arithmetic-mean-state of n qubit density matrices being randomly chosen from the Hilbert–Schmidt ensemble is never decreasing with n. As further illumination of our results, we make an attempt to explain why 'mixing reduces coherence'. We also work out the exact value of the average squared fidelity, studied intensively by K. Życzkowski. Finally, we conclude this paper with some remarks and open problems.

2. Preliminaries

To begin with, we recall some facts about a qubit. Any qubit density matrix can be represented as

$$\rho(\boldsymbol{r}) = \frac{1}{2}(\mathbb{I}_2 + \boldsymbol{r}\cdot\boldsymbol{\sigma}), \tag{2.1}$$

where $\boldsymbol{r} = (r_x, r_y, r_z) \in \mathbb{R}^3$ is the Bloch vector with $r := |\boldsymbol{r}| \leqslant 1$, and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. Here

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are three Pauli matrices. The eigenvalues of a qubit density matrix are given by: $\lambda_\pm(\rho) = \frac{1}{2}(1 \pm r)$, where $r \in [0, 1]$. This leads to the von Neumann entropy of the qubit $\rho(\boldsymbol{r})$ of Bloch vector $\boldsymbol{r}$:

$$\mathrm{S}(\rho(\boldsymbol{r})) = \mathrm{H}_2\left(\frac{1-r}{2}\right) := \Phi(r), \tag{2.2}$$

where $\mathrm{H}_2(p) = -p\log_2 p - (1-p)\log_2(1-p)$ for $p \in [0, 1]$ is the binary entropy function.

Note that the maximal eigenvalue $\lambda_+(\rho)$ for a random qubit density matrix, induced from taking partial trace over a Haar-distributed pure two-qubit state, is subject to the following distribution [11]:

$$\mathrm{d}P(x) = 24\left(x - \frac{1}{2}\right)^2 \mathrm{d}x, \tag{2.3}$$

where $x := \lambda_+(\rho) \in [1/2, 1]$. Since $\lambda_+(\rho) = \frac{1}{2}(1+r)$, it follows that the probability density of the length r of Bloch vector of a random qubit $\rho(\boldsymbol{r})$ is summarized into the following proposition.

Proposition 2.1. *The probability density for the length r of the Bloch vector $\boldsymbol{r}$ in the Bloch representation (2.1) of a random qubit ρ by partial-tracing other qubit system over a Haar-distributed pure two-qubit state, is given by*

$$p_r(r) = 3r^2, \quad r \in [0, 1]. \tag{2.4}$$

Furthermore, the probability distribution of Bloch vector $\boldsymbol{r}$ is given by the formula: $p(\boldsymbol{r})[\mathrm{d}\boldsymbol{r}] = 3r^2\mathrm{d}r \times \frac{1}{4\pi}\delta(1 - |\boldsymbol{u}|)[\mathrm{d}\boldsymbol{u}]$, where δ is the Dirac delta function and $[\mathrm{d}\boldsymbol{u}]$ is the Lebesgue volume element in $\mathbb{R}^3$.

3. Main results

3.1. The spectral density of equiprobable mixture of two qubit states

For $w \in [0, 1]$, we have the probabilistic mixture of two density matrices in a two-level system: $\rho_w(\boldsymbol{r}) = w\rho(\boldsymbol{r}_1) + (1-w)\rho(\boldsymbol{r}_2)$. In particular, for $w = 1/2$, we have the equiprobable mixture, that is, $\rho(\boldsymbol{r}) = \frac{\rho(\boldsymbol{r}_1)+\rho(\boldsymbol{r}_2)}{2}$, hence $\boldsymbol{r} = \frac{\boldsymbol{r}_1+\boldsymbol{r}_2}{2}$. As in [17], denote $\mu, \nu \in [0, 1/2]$ the minimal eigenvalues of two qubit states $\rho(\boldsymbol{r}_1)$ and $\rho(\boldsymbol{r}_2)$, respectively. Then we have $\mu = \frac{1-r_1}{2}$ and $\nu = \frac{1-r_2}{2}$ for $r_1, r_2 \in [0, 1]$. Denote $\mathcal{O}_\mu := \left\{U\mathrm{diag}(1-\mu, \mu)U^\dagger : U \in \mathrm{SU}(2)\right\}$. We consider the equiprobable mixture of two random density matrices $\rho(\boldsymbol{r}_1) \in \mathcal{O}_\mu$ and $\rho(\boldsymbol{r}_2) \in \mathcal{O}_\nu$. In [17], we have already derived the analytical formula for the spectral density of such equiprobable mixture. This result can be summarized into the following proposition.

Proposition 3.1 *([17]). The probability density function of an eigenvalue λ of the equiprobable mixture of two random density matrices, chosen uniformly from respective unitary orbits $\mathcal{O}_\mu$ and $\mathcal{O}_\nu$ with μ, ν are fixed in $(0, 1/2)$, is given by*

$$p(\lambda|\mu, \nu) = \frac{\left|\lambda - \frac{1}{2}\right|}{\left(\frac{1}{2} - \mu\right)\left(\frac{1}{2} - \nu\right)}, \tag{3.1}$$

where $\lambda \in [T_0, T_1] \cup [1 - T_1, 1 - T_0]$. Here $T_0 := \frac{\mu+\nu}{2}$ and $T_1 := \frac{1-|\mu-\nu|}{2}$.

Note that $\lambda \in [T_0, T_1] \cup [1 - T_1, 1 - T_0]$ indicates that the domain of an eigenvalue of the equiprobable mixture: $\frac{1}{2}(U\mathrm{diag}(1-\mu, \mu)U^\dagger + V\mathrm{diag}(1-\nu, \nu)V^\dagger)$, where $U, V \in \mathrm{SU}(2)$.

Given two random density matrices $\rho(\boldsymbol{r}_1) \in \mathcal{O}_\mu$ and $\rho(\boldsymbol{r}_2) \in \mathcal{O}_\nu$. We also see that the eigenvalues of the mixture $\rho(\boldsymbol{r})$ are given by $\lambda = \frac{1\pm r}{2}$. The sign $\pm$ depends on the relationship between λ and $1/2$. Indeed, $\lambda = \frac{1+r}{2}$ if $\lambda \geqslant 1/2$; $\lambda = \frac{1-r}{2}$ if $\lambda \leqslant 1/2$. By using the triple (r_1, r_2, r) instead of (μ, ν, λ), we have the following result.

Theorem 3.2. *The conditional probability density function of the length r of the Bloch vector $\boldsymbol{r}$ of the equiprobable mixture: $\rho(\boldsymbol{r}) = \frac{\rho(\boldsymbol{r}_1)+\rho(\boldsymbol{r}_2)}{2}$, where $r_1, r_2 \in (0, 1)$ are fixed, is given by*

$$p(r|r_1, r_2) = \frac{2r}{r_1 r_2}, \tag{3.2}$$

where $r \in [r_-, r_+]$ with $r_- := \frac{|r_1-r_2|}{2}$ and $r_+ := \frac{r_1+r_2}{2}$.

Denote by θ the angle between Bloch vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ from two random density matrices $\rho(\boldsymbol{r}_1) \in \mathcal{O}_\mu$ and $\rho(\boldsymbol{r}_2) \in \mathcal{O}_\nu$, respectively. Apparently $\theta \in [0, \pi]$. Since $\rho(\boldsymbol{r}) = \frac{\rho(\boldsymbol{r}_1)+\rho(\boldsymbol{r}_2)}{2}$, i.e., $\boldsymbol{r} = \frac{\boldsymbol{r}_1+\boldsymbol{r}_2}{2}$, it follows that $r = \frac{1}{2}\sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos\theta}$, where $\theta \in [0, \pi]$.

Clearly the rhs is the invertible function of the argument θ defined over $[0, \pi]$ when r_1 and r_2 are fixed. In view of this, we see that the angle between two random Bloch vectors has the following probability density:

$$f(\theta) = \frac{1}{2}\sin\theta, \quad \theta \in [0, \pi]. \tag{3.3}$$

3.2. The quantum addition rule for two qubit states

Shannon's Entropy Power Inequality mainly deals with the concavity of an entropy function of a continuous random variable under the scaled addition rule: $f\left(\sqrt{w}X + \sqrt{1-w}Y\right) \geqslant wf(X) + (1-w)f(Y)$, where $w \in [0, 1]$ and X, Y are continuous random