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A new construction of rational electromagnetic knots

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ABSTRACT

We set up a correspondence between solutions of the Yang–Mills equations on $\mathbb{R} \times S^3$ and in Minkowski spacetime via de Sitter space. Some known Abelian and non-Abelian exact solutions are rederived. For the Maxwell case we present a straightforward algorithm to generate an infinite number of explicit solutions, with fields and potentials in Minkowski coordinates given by rational functions of increasing complexity. We illustrate our method with a nontrivial example.

1. Conformal equivalence of dS_4 to $\mathcal{I} \times S^3$ and two copies of $\mathbb{R}^{1,3}_+$

The present work is motivated by the recent paper [1] co-authored by one of us, where analytic solutions of the Yang–Mills equations on four-dimensional de Sitter space dS_4 are constructed. It is well known that de Sitter space can be realized as the single-sheeted hyperboloid

$$-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 \;=\; \ell^2 \tag{1.1}$$

embedded in five-dimensional Minkowski space $\mathbb{R}^{1,4}$ with the metric

$$ds^2 \;=\; -dZ_0^2 + dZ_1^2 + dZ_2^2 + dZ_3^2 + dZ_4^2 \,. \tag{1.2}$$

Constant Z_0 slices of the hyperboloid reveal a three-sphere of varying radius. The following parametrization makes this structure explicit:

$$Z_0 \;=\; -\ell \cot\tau \qquad \text{and} \qquad Z_A \;=\; \frac{\ell}{\sin\tau}\,\omega_A \quad \text{for} \quad A=1,\dots,4\,, \tag{1.3}$$

where the coordinates ω_A embed a unit three-sphere into R^4, and $0<\tau<\pi$, i.e.

$$\omega_A\omega_A = 1 \qquad \text{and} \qquad \tau\in\mathcal{I} := (0,\pi)\,. \tag{1.4}$$

The metric of dS_4 in such coordinates becomes

$$ds^2 \;=\; \frac{\ell^2}{\sin^2\tau}\bigl(-d\tau^2 + d\Omega_3^2\bigr)\,, \tag{1.5}$$

where $d\Omega_3^2$ denotes the metric of the unit three-sphere. Hence, four-dimensional de Sitter space is conformally equivalent to a finite Minkowskian cylinder over a three-sphere.

Part of it is also conformally equivalent to (half of) Minkowski space, by employing the parametrization

$$Z_0 \;=\; \frac{t^2-r^2-\ell^2}{2t}\,, \quad Z_1 \;=\; \ell\,\frac{x}{t}\,, \quad Z_2 \;=\; \ell\,\frac{y}{t}\,, \quad Z_3 \;=\; \ell\,\frac{z}{t}\,, \quad Z_4 \;=\; \frac{r^2-t^2-\ell^2}{2t}\,, \tag{1.6}$$

where

$$x,y,z\in\mathbb{R} \qquad \text{and} \qquad r^2=x^2+y^2+z^2 \qquad \text{but} \qquad t\in\mathbb{R}_+ \tag{1.7}$$

since $t\to 0$ corresponds to $Z_0\to-\infty$. The metric of dS_4 becomes

$$ds^2 \;=\; \frac{\ell^2}{t^2}\bigl(-dt^2 + dx^2 + dy^2 + dz^2\bigr)\,, \tag{1.8}$$

hence these coordinates cover the future half $\mathbb{R}^{1,3}_+$ of Minkowski space. In a moment this parametrization will be extended to the whole of Minkowski space, by gluing a second copy of dS_4 to provide for the $t<0$ half. The de Sitter radius ℓ provides a scale.

We shall need the direct relation between the cylinder and Minkowski coordinates. By comparing (1.3) and (1.6) we see that

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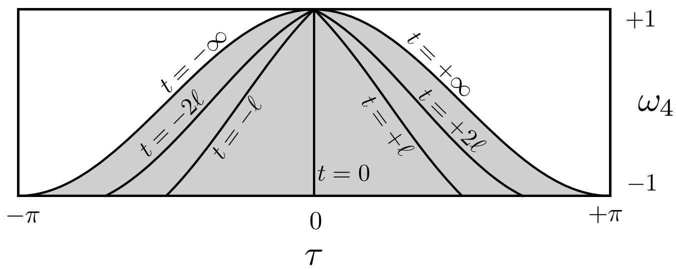

Fig. 1. An illustration of the map between a cylinder $2\mathcal{I}\times S^3$ and Minkowski space $R^{1,3}$. The Minkowski coordinates cover the shaded area. The boundary of this area is given by the curve $\omega_4=\cos\tau$. Each point is a two-sphere spanned by $\omega_{1,2,3}$, which is mapped to a sphere of constant r and t.

$$
\begin{aligned}
-\cot\tau \;&=\; \frac{t^2-r^2-\ell^2}{2\,\ell\,t}\,, \quad \omega_1 \;=\; \gamma\,\frac{x}{\ell}\,, \quad \omega_2 \;=\; \gamma\,\frac{y}{\ell}\,, \\
\omega_3 \;&=\; \gamma\,\frac{z}{\ell}\,, \quad \omega_4 \;=\; \gamma\,\frac{r^2-t^2-\ell^2}{2\,\ell^2}\,,
\end{aligned}
\tag{1.9}
$$

where for convenience we abbreviated the frequent combination

$$
\gamma \;=\; \frac{2\,\ell^2}{\sqrt{4\,\ell^2 t^2+(r^2-t^2+\ell^2)^2}}\,. \tag{1.10}
$$

If we fix r and let t vary from $-\infty$ to ∞, then $-\cot\tau$ sweeps two branches. We pick the branches so that $\tau\in(-\pi,0)$ for $t<0$ and $\tau\in(0,\pi)$ for $t>0$, gluing them at $\tau=t=0$. Then inverting (1.9) produces τ as a regular function of (t,x,y,z). A more useful relation for the following is

$$
\exp(\mathrm{i}\,\tau) \;=\; \frac{(\ell+\mathrm{i}t)^2+r^2}{\sqrt{4\,\ell^2 t^2+(r^2-t^2+\ell^2)^2}}\,. \tag{1.11}
$$

Hence, comparing (1.5) and (1.8), we have given an explicit conformal equivalence between full Minkowski space $\mathbb{R}^{1,3}$ and a patch of a finite S^3-cylinder $2\mathcal{I}\times S^3$ with $2\mathcal{I}=(-\pi,\pi)\ni\tau$. The structure of this equivalence is best clarified by an illustration (see Fig. 1). Note that the whole infinite $R\times S^3$ cylinder can be covered by such patches. The neighboring patches can be related via shifting τ by π and changing the sign of ω_4. The latter action essentially implements a parity transformation.

2. The correspondence

In four spacetime dimensions Yang–Mills theory is conformally invariant. Therefore, instead of solving its equations of motion on Minkowski space one may solve them on the cylinder $2\mathcal{I}\times S^3$. The latter has the added advantage yielding a manifestly SO(4)-covariant formalism due to the three-sphere. Furthermore, S^3 is the group manifold of SU(2), which enables the geometric parametrization (we pick the temporal gauge $A_\tau=0$)

$$
A \;=\; \sum_{a=1}^{3} X_a(\tau,\omega)\,e^a\,, \tag{2.1}
$$

where X_a are three functions of τ and $\omega\equiv\{\omega_A\}$ valued in some Lie algebra, and e^a are the three left-invariant one-forms on S^3. Since the conformal factor is irrelevant for the Yang–Mills equations we can translate Yang–Mills solutions on $2\mathcal{I}\times S^3$ to solutions on $\mathbb{R}^{1,3}$ simply via a change of coordinates. The behavior at the boundary $\cos\tau=\omega_4$ is thereby transferred to fall-off properties at temporal infinity $t\to\pm\infty$.

To become explicit, we need Minkowski-coordinate expressions for the one-forms $e^0=\mathrm{d}\tau$ and e^a, which are subject to

$$
\mathrm{d}e^a+\varepsilon^a_{\ bc}\,e^b\wedge e^c \;=\; 0 \qquad\text{and}\qquad e^a e^a \;=\; \mathrm{d}\Omega_3^2\,. \tag{2.2}
$$

They can be constructed as

$$
e^a \;=\; -\eta^a_{BC}\,\omega_B\,\mathrm{d}\omega_C\,, \tag{2.3}
$$

with η^a_{BC} denoting the self-dual 't Hooft symbol (with non-zero components $\eta^i_{jk}=\varepsilon^i_{\ jk}$ and $\eta^i_{j4}=-\eta^i_{4j}=\delta^i_j$). A straightforward computation yields ($a,j,k=1,2,3$)

$$
\begin{aligned}
e^0 \;&=\; \frac{\gamma^2}{\ell^3}\Big(\tfrac12(t^2+r^2+\ell^2)\,\mathrm{d}t-t\,x^k\mathrm{d}x^k\Big)\,, \\
e^a \;&=\; \frac{\gamma^2}{\ell^3}\Big(t\,x^a\mathrm{d}t-\big(\tfrac12(t^2-r^2+\ell^2)\,\delta^a_k+x^a x^k+\ell\,\varepsilon^a_{\ jk}x^j\big)\,\mathrm{d}x^k\Big)\,,
\end{aligned}
\tag{2.4}
$$

where we introduce the standard notation

$$
(x^i)=(x,y,z) \qquad\text{and (for later)}\qquad (x^\mu)=(x^0,x^i)=(t,x,y,z)\,. \tag{2.5}
$$

Two remarks are in order. First, in Minkowski spacetime the parameter ℓ just sets an overall scale, which is needed for nontrivial solutions because the Yang–Mills equations themselves are scale-invariant in four dimensions. Second, at fixed t the components for $e^0,\dots,e^3$ decay at least as $1/r^2$ for large r. This is a good signal that the solutions translated from the cylinder will have finite energy in $\mathbb{R}^{1,3}$.

Let us see how this works by transferring some solutions obtained in [1,2] to Minkowski spacetime.[1] There, the authors restricted to SO(4)-symmetric configurations by taking $X_a=X_a(\tau)$ to be independent of ω. This ansatz reduces the Yang–Mills equations to ordinary differential equations for the matrices X_a. On the cylinder, a simple static homogeneous solution is given by

$$
\begin{aligned}
X_a(\tau) \;&=\; \tfrac12\,T_a \qquad\Rightarrow \\
A \;&=\; \tfrac12\,g^{-1}\mathrm{d}g \qquad\text{for}\quad g: S^3\to \mathrm{SU}(2)\,,
\end{aligned}
\tag{2.6}
$$

where T_a are $su(2)$ algebra generators scaled to obey $[T_a,T_b]=2\varepsilon_{abc}T_c$. After inserting (2.4) and (2.6) into the ansatz (2.1) one recognizes the De Alfaro–Fubini–Furlan solution [3] (see also [4]). A more general case,

$$
X_a(\tau) \;=\; \big(1+\tfrac12 q(\tau)\big)\,T_a \qquad\text{with}\qquad \frac{\mathrm{d}^2q}{\mathrm{d}\tau^2} \;=\; -\frac{\partial V}{\partial q} \qquad \text{for}\quad V(q) \;=\; \tfrac12 q^2(q+2)^2\,, \tag{2.7}
$$

produces a family of SO(4)-symmetric solutions studied by Lüscher [5]. For a review on analytic Yang–Mills solutions, see [6].

However, the interest of this paper is in Abelian solutions, i.e. electromagnetic field configurations. These may be embedded in the non-Abelian framework by demanding the three matrices X_a to all be proportional to the same fixed Lie-algebra element, say T_3. Such solutions on $2\mathcal{I}\times S^3$ (with two proportionality coefficients vanishing) were also discussed in [2]. Since in the U(1) case the matrix structure is irrelevant, from now on we take $X_a(\tau,\omega)$ simply to be real-valued functions and focus on Maxwell's equations. In the SO(4)-invariant case, $X_a=X_a(\tau)$ are found to obey the oscillator equation

$$
\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}X_a(\tau) \;=\; -4\,X_a(\tau) \quad\Rightarrow\quad X_a(\tau) \;=\; c_a\,\cos\big(2(\tau-\tau_a)\big)\,, \tag{2.8}
$$

yielding six integration constants in the general solution. Since the X_a are oscillating with a frequency of two, we can use the square

[1] In these papers cylinder solutions were transferred to dS_4 solutions.