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Noteworthy fractal features and transport properties of Cantor tartans

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A B S T R A C T

This Letter is focused on the impact of fractal topology on the transport processes governed by different kinds of random walks on Cantor tartans. We establish that the spectral dimension of the infinitely ramified Cantor tartan d_s is equal to its fractal (self-similarity) dimension D. Consequently, the random walk on the Cantor tartan leads to a normal diffusion. On the other hand, the fractal geometry of Cantor tartans allows for a natural definition of power-law distributions of the waiting times and step lengths of random walkers. These distributions are Lévy stable if $D > 1.5$. Accordingly, we found that the random walk with rests leads to sub-diffusion, whereas the Lévy walk leads to ballistic diffusion. The Lévy walk with rests leads to super-diffusion, if $D > \sqrt{3}$, or sub-diffusion, if $1.5 < D < \sqrt{3}$.

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1. Introduction

Mass and momentum transport are ubiquitous to natural processes, as well as to engineered systems [1–3]. Transport processes in complex systems are strongly dependent on the system topology [4–12]. Accordingly, fractal networks are widely used to model the transport phenomena in heterogeneous media (see, for example, Refs. [1–12] and references therein). Therefore, understanding effects of fractal features on the transport properties is of crucial importance from both scientific and technological standpoints. In this regard, the fractal attributes can be characterized by a set of dimension numbers [12–17]. Specifically, the fractal mass distribution in the embedding Euclidean space is characterized by the fractal (e.g. self-similarity or Hausdorff) dimension D [12]. The fractal connectivity is characterized by the connectivity dimension d_ℓ also called the chemical or spreading dimension [12–14]. Generally, $d_\ell = D/d_{\min}$, where $d_{\min}$ is the fractal dimension of the minimum path between two randomly chosen points on the fractal [12–17]. The order of ramification R at site i of the path-connected fractal is equal to the number of significant bonds which one must cut in order to isolate an arbitrarily large bounded set of points connected to i [18]. For the infinitely ramified fractals this number grows as a power of the size of this bounded set with the scaling exponent Q_i. Accordingly, the ramification is characterized by the connectivity exponent $Q = \min_i\{Q_i\}$, while for finitely ramified fractal $Q = 0$ [18]. In Ref. [10] it was proved that the connectivity exponent of the path-connected fractal is related to its topological Hausdorff dimension D_{tH} as $Q = D_{tH} - 1$, while, generally, $1 \le D_{tH} \le D$ (see Ref. [19]). The number of dynamical degrees of freedom of random walker on the fractal is equal to the spectral dimension d_s [16,20]. The last is commonly defined either via the asymptotic behavior of density of vibrational modes $N \propto \omega^{d_s - 1}$, or by the scaling of probability that the random walker on the fractal returns to the starting point after t steps $P_o \propto t^{d_s/2}$ [12–17]. The mean squared displacement of the random walker scales with time as

$$\langle \delta^2 \rangle \propto t^\gamma, \tag{1}$$

where γ is the diffusion exponent. The normal diffusion is characterized by $\gamma = 1$. If $\gamma \neq 1$, the diffusion is called anomalous. Specifically, the sub-diffusion is characterized by $0 < \gamma < 1$, the super-diffusion is characterized by $1 < \gamma < 2$, and the ballistic diffusion is characterized by $\gamma = 2$ [14]. The ratio $D_W = 2/\gamma$ is called the fractal dimension of random walk [12–17]. The classical Brownian motion is characterized by $D_W = 2$, whereas the random walk on path-connected fractals is characterized by $D_W \ge 2$ [12–19]. Accordingly, the diffusion on path-connected fractals becomes a key paradigm of the sub-diffusion [21–24]. For finitely

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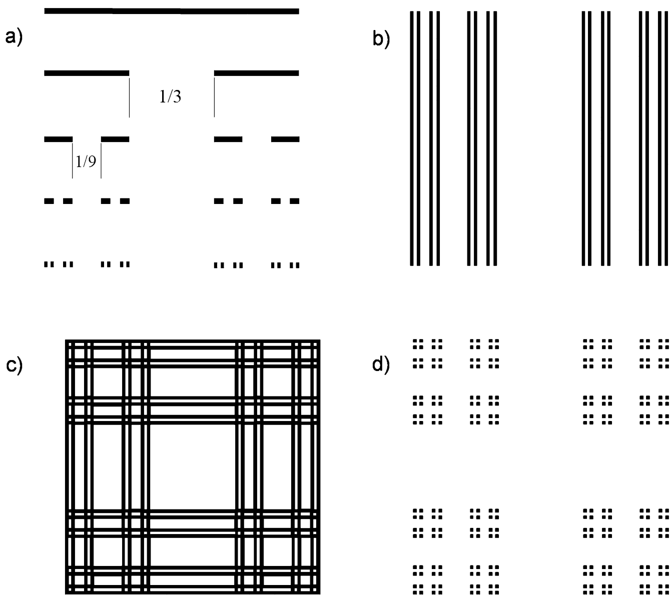

Fig. 1. Construction Cantor tartan: a) first four iterations of the ternary Cantor set C_3^1 ($\alpha = \ln 2/\ln 3$); b) Cartesian product $C_3^1 \times [0,1]$; c) Cantor tartan $T_3^1 = C_3^1 \times [0,1] \cup C_3^1 \times [1,0]$ with the fractal dimension $D = 1 + \alpha = \ln 6/\ln 3$; d) the intersection $C_3^1 \times [0,1] \cap C_3^1 \times [1,0] = C_3^1 \times C_3^1$ with the fractal dimension $2\alpha = \ln 4/\ln 3$.

ramified fractals, there are some well developed theoretical tools concerning spectral asymptotics [23–28]. This allows to calculate the spectral dimension of the finitely ramified fractal exactly. However, yet there are no similar results for infinitely ramified fractals. Accordingly, all known values of d_s for the infinitely ramified fractals were obtained from numerical simulations.

In this work, we found that for a special class of infinitely ramified fractals, called the Cantor tartans (see Ref. [29]), the spectral dimension d_s is equal to the fractal (box-counting) dimension D, while the connectivity exponent is equal to $Q = D_{tH} - 1 = D - 1$. Therefore, the random walk on the Cantor tartan leads to the normal diffusion. On the other hand, the fractal geometry of Cantor tartans allows for a natural definition of power-law distributions of the waiting times and step lengths of random walkers. Accordingly, different kinds of random walks on the Cantor tartans lead to different types of anomalous diffusion with the diffusion exponent determined by D. The rest of the paper is organized as follows. The fractal features of Cantor tartans are discussed in Section 2. In Section 3 we prove that the spectral dimension of the Cantor tartan is equal to its fractal dimension. The scaling behavior of electrical resistance and absolute permeability of Cantor tartans are established. Section 4 is devoted to study of different kinds of random walks on the Cantor tartans. The relevant conclusions are outlined in Section 5.

2. Fractal features of Cantor tartans

Cantor tartan is a union of two Cartesian products $C_y \times [1,0]$ and $C_x \times [0,1]$, such that the intersection between these products is a totally disconnected Cantor set $C_{xy} = C_x \times C_y$ on a plane $[1,0] \times [0,1]$, while $C_x \subset [1,0]$ and $C_y \subset [0,1]$ are the Cantor sets on real line (see Fig. 1). The ternary Cantor set is a classical example of a perfect nowhere-dense set on the real line [30]. Geometrically, it is constructed by iterative deletion of open middle-third intervals from remaining intervals of the previous iteration, starting from the unit interval $[0,1]$ *ad* infinitum (see Fig. 1a). The fractal dimension of the ternary Cantor is equal to its similarity dimension $\alpha = \ln 2/\ln 3$. A generalized Cantor set with the fractal dimension in the range of $0 < \alpha < 1$ can be constructed in a similar manner [31]. It has been proved that every Cantor set is homeomorphic to the ternary Cantor set and any compact metric space is a continuous image of the ternary Cantor set [32]. Due to these remarkable properties the totally disconnected Cantor sets have found a celebrated place in mathematical analysis and its applications (see Refs. [30–36] and references therein). However, obviously, there is no way to build a nontrivial Markov process having continuous trajectories on the totally disconnected Cantor set embedded in the Euclidean space. Accordingly, the random walks defined on the Cantor set are dependent on how the continuum requirement is handled [37–41]. In contrast to this, the Cantor tartans are totally connected infinitely ramified fractals. The fractal dimension of the Cartesian product $C \times [1,0]$ (see Fig. 1b) is equal to $D = 1 + \alpha$, while the fractal dimension of the union of finite number of fractals with the fractal dimensions D^i is equal to $D = \max_i\{D^i\}$ [42]. So, the fractal dimension of the Cantor tartan (see Fig. 1c) is equal to

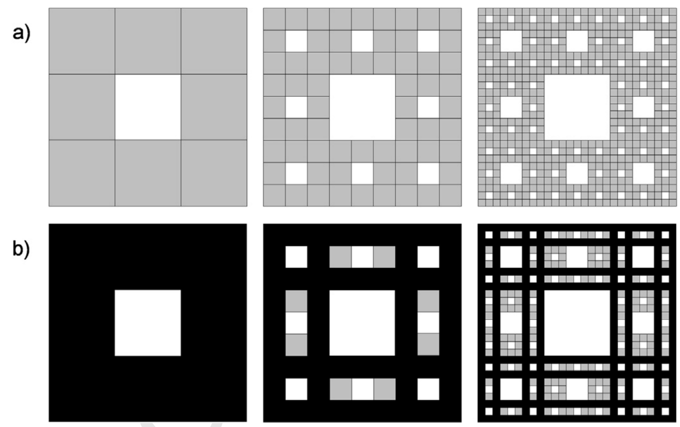

Fig. 2. First three iteration of the construction of: a) Sierpiński carpet S_3^1 with $D(S_3^1) = \ln 8/\ln 3$, $D_{tH}(S_3^1) = \ln 6/\ln 3$, and $Z_\infty(S_3^1) = 3.2$; and b) Cantor tartan T_3^1 imbedded into Sierpiński carpet S_3^1. The number of square boxes with edge size $l_k = 3^{-k}$ needed to cover the Cantor tartan after k iterations is equal to $N_k = 2(2^k 3^k) - 2^{2k}$, so that the fractal (box-counting) dimension $D = -\lim_{k\to\infty}(\ln N_k/\ln l_k) = 1 + \ln 2/\ln 3$ is equal to the fractal (self-similarity) dimension $D = 1 + \alpha < D(S_3^1)$, while $D_{tH} = D = D_{tH}(S_3^1)$ and $Z_\infty(T_3^1) = 2$.

$$D = 1 + \alpha, \tag{2}$$

whereas the intersection between $C_y \times [1,0]$ and $C_x \times [0,1]$ (see Fig. 1d) has the fractal dimension $D^{int} = 2\alpha$. In three dimensional space E^3 the Cantor tartan with the fractal dimension $D = 1 + 2\alpha$ can be constructed as a union of three orthogonal Cartesian products $(C_x \times C_y) \times [0,0,1]$, $(C_x \times C_z) \times [0,1,0]$, and $(C_y \times C_z) \times [1,0,0]$.

Alternatively, the Cantor tartan can be constructed iteratively as a generalized Sierpiński carpet (see Fig. 2). A standard Sierpiński carpet S_n^m is obtained by iterative removing the interior of the central segment of size m^2 from each group of n^2 subsquares, while $m \le n - 2$ (see Fig. 2a). It is a straightforward matter to see that the Cantor tartan T_n^m is a subset of the standard Sierpiński carpet S_n^m (see Fig. 2b and Ref. [10] for more details). Accordingly, before analyzing the fractal features of Cantor tartans, let us outline the fractal attributes of generalized Sierpiński carpets $S_n^{m,\kappa}$ constructed by removing m^2 subsquares which form κ^2 separated subarrays interior the group of n^2 subsquares (see Ref. [10]).

On the Sierpiński carpet $S_n^{m,\kappa}$ there are three kinds of sites characterized by the local coordination numbers (numbers of nearest neighborhoods) equal to 2, 3, and 4, respectively (see, for example, Fig. 2a). The total number of sites on the Sierpiński carpet increases with the number of iteration steps k as $N_k = 3^{Dk}$, where

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