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How entangled can a multi-party system possibly be?

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ABSTRACT

The geometric measure of entanglement of a pure quantum state is defined to be its distance to the space of pure product (separable) states. Given an n-partite system composed of subsystems of dimensions $d_1, \ldots, d_n$, an upper bound for maximally allowable entanglement is derived in terms of geometric measure of entanglement. This upper bound is characterized exclusively by the dimensions $d_1, \ldots, d_n$ of composite subsystems. Numerous examples demonstrate that the upper bound appears to be reasonably tight.

1. Introduction

The physical realization of quantum computing calls for a hierarchical quantum network. The bottom level is the one- and two-qubit regime, where a photon interacts with matter (e.g., a trapped ion). In this regime, precise control must be exerted. Going one level up we enter the regime of quantum logic gates where typically ten or more qubits operate. One level further up is the fault-tolerant quantum error correction (QEC) architecture regime where hundreds of qubits reside. The final level is the algorithms regime. Being an essential resource for quantum computing, entanglement propagates over the dynamic quantum network to fulfill desired quantum computing tasks. A fundamental question naturally arises: how much entanglement can a quantum network encode?

If a quantum network is composed of qubits, that is, each particle lives in a two-dimensional Hilbert space, we end up with a multipartite qubit system. When restricted to the pure state case, the 2-qubit entanglement is well-understood. For the 3-qubit case, it is well-known that the GHZ state [1] is the most entangled state in terms of entanglement entropy and its degree of entanglement can also be easily computed by means of many other measures of entanglement. On the other hand, it has been reported [2] that the 3-qubit W-state [3] is more entangled than the 3-qubit GHZ state in terms of geometric measure of entanglement [4]. In fact, it is generally agreed that the characterization and quantification of entanglement of n-qubit systems for $n > 3$ is a difficult task. In addition to qubits, for a typical quantum network, there may also exist other finite-level units, see several recent experimental set-ups in e.g., [5–8]. For such a hybrid quantum network, namely a heterogeneous multipartite system, it is unclear how much entanglement can be allowed, not to mention how to quantify it efficiently.

A fundamental problem in quantum physics and also an important problem in quantum information science is to detect whether a given state is entangled, and if so, how entangled it is. Several measures of quantum entanglement have already been proposed in the literature, e.g., Schmidt rank [9, Section 2.5], von Neumann entropy [9, Section 11.3], entanglement of formation [10], quantum concurrence [11,12], the Peres–Horodecki criterion [13, 14]. Schmidt measure (also called Hartley entropy) [15] based on Candecomp/Parafac (CP) decomposition of tensors [16,17], relative entropy [18], negativity [19], the geometric measure of entanglement, [4,20–30]. More can be found in the survey papers [31–33]. For the bipartite pure state case, a state is maximally entangled in terms of one measure is often also maximally entangled in terms of another measure. In this sense, different measures give consistent prediction. This is not true for multipartite cases. For a multipartite system, it is typical that two different measures attain their maxima at different quantum states [3,34].

In this paper we are interested in the following problem: Given an n-partite system which can be either homogeneous or heterogeneous, how entangled can its states be? We will use the geometric measure to quantify the degree of entanglement. We show that an upper bound can be derived for entanglement content allowed. Moreover, the upper bound is given exclusively in terms of dimensions of the composite subsystems. Not surprisingly, the up-

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per bound can always be reached in the case of bipartite systems. Interestingly, various examples demonstrate that upper bounds appear to be reasonably tight for many multipartite systems.

2. Geometric measure of entanglement (GME)

For a quantum n-partite system, a pure state $|\Psi\rangle$ is an element in the tensor product Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \equiv \otimes_{k=1}^n \mathcal{H}_k$. For each $k = 1, \ldots, n$, denote the dimension of the composite subsystem $\mathcal{H}_k$ by d_k and the orthonormal basis by $\{|e_{i_k}^{(k)}\rangle : i_k = 1, \ldots, d_k\}$. For ease of presentation and without loss of generality, it is assumed in this paper that

$$d_1 \le d_2 \le \cdots \le d_n.$$

A pure state $|\Psi\rangle \in \mathcal{H}$ is of the from

$$|\Psi\rangle = \sum \bar{a}_{i_1 \cdots i_n} |e_{i_1}^{(1)}\rangle \otimes \cdots \otimes |e_{i_n}^{(n)}\rangle, \tag{1}$$

where $\bar{a}_{i_1 \cdots i_n} \in \mathbb{C}$ (the "bar" stands for the complex conjugation). The normalization condition of $|\Psi\rangle$ is $\| |\Psi\rangle \|^2 \equiv \langle \Psi | \Psi \rangle = \sum_{i_1, \cdots, i_n} |a_{i_1 \cdots i_n}|^2 = 1$. A state $|\phi\rangle \in \mathcal{H}$ is said to be *separable* if it is a product state

$$|\phi\rangle = |\phi^{(1)}\rangle \otimes \cdots \otimes |\phi^{(n)}\rangle, \tag{2}$$

where

$$|\phi^{(k)}\rangle = \sum u_{i_k}^{(k)} |e_{i_k}^{(k)}\rangle \in \mathcal{H}_k, \quad \forall k = 1, \ldots, n. \tag{3}$$

If a state is not separable, then it is called an *entangled* state.

Next, let us briefly review the geometric measure of entanglement, more details can be found in, e.g., [4,20–30] and references therein. Denote the set of all separable pure states in $\mathcal{H}$ as $Separ(\mathcal{H})$. For a general n-partite state $|\Psi\rangle \in \mathcal{H}$, the geometric measure of its entanglement content can be defined as its distance to the space of separable states $Separ(\mathcal{H})$, [4,29], i.e.,

$$\mathrm{GME}_\Psi \triangleq \min \{ \| |\Psi\rangle - |\phi\rangle \| : |\phi\rangle \in Separ(\mathcal{H}) \}. \tag{4}$$

Since the minimization in (4) is taken with a continuous function on a compact set $Separ(\mathcal{H})$ in a finite dimensional space $\mathcal{H}$, the minimizer does exist and is denoted by $|\phi_\Psi\rangle \in Separ(\mathcal{H})$. Clearly, $|\phi_\Psi\rangle$ is the separable state which is closest to $|\Psi\rangle$.

For convenience, as in [4,29], instead of computing (4) directly, we study

$$\begin{aligned} \mathrm{GME}_\Psi^2 &= \| |\Psi\rangle - |\phi_\Psi\rangle \|^2 \\ &= \min \left\{ \| |\Psi\rangle - |\phi\rangle \|^2 : |\phi\rangle \in Separ(\mathcal{H}) \right\}. \end{aligned} \tag{5}$$

Note that

$$\| |\Psi\rangle - |\phi\rangle \|^2 = 2 - \langle \Psi | \phi \rangle - \langle \phi | \Psi \rangle.$$

Thus the minimization problem in (5) is equivalent to the following maximization problem:

$$\max_{\langle \phi^{(k)} | \phi^{(k)} \rangle = 1, k = 1, \cdots, n} \left\{ \langle \Psi | \otimes_{k=1}^n \phi^{(k)} \rangle + \otimes_{k=1}^n \langle \phi^{(k)} | \Psi \rangle \right\}. \tag{6}$$

Introducing Lagrange multipliers λ_k, $k = 1, \cdots, n$, and applying complex differentiation [35] to get

$$\langle \Psi | \otimes_{j=1, j \ne k}^n |\phi^{(j)}\rangle = \lambda_k \langle \phi^{(k)}|,$$

and

$$\otimes_{j=1, j \ne k}^n \langle \phi^{(j)} | \Psi \rangle = \lambda_k |\phi^{(k)}\rangle.$$

Therefore,

$$\lambda_k = \langle \Psi | \phi \rangle = \langle \phi | \Psi \rangle, \quad k = 1, \ldots, n$$

is a *real* number in the interval $[-1, 1]$. Denote the maximal overlap by, [4],

$$\langle \Psi | \phi_\Psi \rangle \triangleq \max \{ | \langle \Psi | \phi \rangle | : |\phi\rangle \in Separ(\mathcal{H}) \}, \tag{7}$$

and the geometric measure of entanglement of the pure state $|\Psi\rangle$, defined in (4), is hence

$$\mathrm{GME}_\Psi = \sqrt{2 - 2 \langle \Psi | \phi_\Psi \rangle}. \tag{8}$$

Clearly, the smaller the maximal overlap $\langle \Psi | \phi_\Psi \rangle$ is, the bigger the distance GME_Ψ between $|\Psi\rangle$ and the set of separable states.

Next, we represent the geometric measure of entanglement in terms of tensor (also called hypermatrix) [16,17]. For the pure state $|\Psi\rangle$ in (1), we define an associated tensor $\mathcal{A}_\Psi$ by $\mathcal{A}_\Psi = (a_{i_1 \cdots i_n}) \in \mathbb{C}^{d_1 \times \cdots \times d_n}$. That is, we store all the probability amplitudes of the state $|\Psi\rangle$ into a multi-array. Similarly, we associate each $|\phi^{(k)}\rangle$ in (3) with a column vector $u^{(k)} \in \mathbb{C}^{d_k}$, $k = 1, \ldots, n$. Then we define a c-number

$$\mathcal{A}_\Psi u^{(1)} \cdots u^{(n)} \triangleq \sum a_{i_1 \cdots i_n} u_{i_1}^{(1)} \cdots u_{i_n}^{(n)}. \tag{9}$$

With this notation, the inner product between $|\Psi\rangle$ in (1) and $|\phi\rangle$ in (2) can be re-written as

$$\langle \Psi | \phi \rangle = \mathcal{A}_\Psi u^{(1)} \cdots u^{(n)}. \tag{10}$$

Denote the spectral radius of the tensor $\mathcal{A}$ by

$$\sigma(\mathcal{A}_\Psi) \triangleq \max_{\|u^{(k)}\|^2 = 1, \, k = 1, \cdots, n} |\mathcal{A}_\Psi u^{(1)} \cdots u^{(n)}|. \tag{11}$$

(It is worth noting that when $n = 2$, the tensor $\mathcal{A}_\Psi$ reduces to a $d_1 \times d_2$ matrix. In this case, $\sigma(\mathcal{A}_\Psi)$ is actually the largest singular value of the matrix $\mathcal{A}_\Psi$.) Then the largest overlap in (7) can be expressed as

$$\langle \Psi | \phi_\Psi \rangle = \sigma(\mathcal{A}_\Psi). \tag{12}$$

As a result, the geometric measure of entanglement of the multipartite state $|\Psi\rangle$, expressed in (8), becomes

$$\mathrm{GME}_\Psi = \sqrt{2 - 2\sigma(\mathcal{A}_\Psi)}. \tag{13}$$

In the literature of tensor optimization, several algorithms have been developed for computing the spectral radius of a given tensor $\mathcal{A}$. When $\mathcal{A}$ is symmetric, it can be proved that the spectral radius can be obtained when $u^{(1)} = \cdots = u^{(n)}$, [36]. In particular, if further $\mathcal{A}$ is real and with all nonnegative entries, then the spectral radius is given by its largest Z-eigenvalue [29,37]. In general, the spectral radius of a symmetric tensor $\mathcal{A}$ is its largest unitary symmetric eigenvalue (US-eigenvalue) [26]. An algorithm has been developed to find the largest US-eigenvalue of a given symmetric tensor [38, Algorithm 4.1]. When $\mathcal{A}$ is non-symmetric, its spectral radius is its largest unitary eigenvalue (U-eigenvalue) [26]. The algorithm proposed in [38] can be modified to find the largest U-eigenvalue of a given non-symmetric tensor, see the algorithm in APPENDIX. All the examples in this paper are computed using these two algorithms.