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Generalization of uncertainty relation for quantum and stochastic systems

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ABSTRACT

The generalized uncertainty relation applicable to quantum and stochastic systems is derived within the stochastic variational method. This relation not only reproduces the well-known inequality in quantum mechanics but also is applicable to the Gross–Pitaevskii equation and the Navier–Stokes–Fourier equation, showing that the finite minimum uncertainty between the position and the momentum is not an inherent property of quantum mechanics but a common feature of stochastic systems. We further discuss the possible implication of the present study in discussing the application of the hydrodynamic picture to microscopic systems, like relativistic heavy-ion collisions.

1. Introduction

Variational formulation is the standard approach to incorporate symmetries of a given system which play fundamental roles in its dynamics. Unfortunately, there exist several cases where such an approach is not applicable. Therefore the extension of the variational principle is worthwhile to be investigated [1].

Let us consider for example the variational formulation of classical mechanics. There, the evolution of a system is described by optimizing the corresponding action with respect to virtual trajectories for which we can define at least the second order time derivative. Therefore, if we extend the domain of the virtual trajectories to include non-differentiable trajectories such as the Brownian motion, we should introduce a new variational approach [2]. Such a generalization of the variational principle is known as the stochastic control problem in the stochastic calculus and there are various works in this direction [1,3–19]. In this paper, we consider the stochastic variational method (SVM) proposed by Yasue [1].

This generalization provides us a possible unified description of classical and quantum behaviors. In fact, we can derive the Schrödinger equation by employing the stochastic variation to the action which leads to the Newton equation under the application of the classical variation. Although the framework of SVM was originally proposed to reformulate Nelson's stochastic quantization [20], its applicability is not restricted to the quantization problem. The Navier–Stokes–Fourier equation is obtain by employing the stochastic variation to the classical action of the Euler (ideal fluid) equation [21]. This method is useful also to introduce the model where the quantum and classical degrees of freedom coexist [22]. It is also worth mentioning that Schrödinger developed a classical probability theory where the probability density is given by the product of real wave functions. This is called reciprocal process [23–33]. In Ref. [27], it is shown that the evolution of the reciprocal process can be formulated in the form of SVM.

Such a generalized perspective enables us to find the correspondence between stochastic and quantum behaviors. For example, there exists the well-known fundamental limitation for simultaneous measurements between two canonical variables in quantum mechanics. This uncertainty principle by Heisenberg constitutes one of the intrinsic features of quantum mechanics. Mathematically, its origin traces back to the non-commutative nature of the operators corresponding to the observables in question. On the other hand, in the framework of SVM, the observables are expressed as not only operators but also stochastic quantities [18,34]. Thus the algorithm to derive the uncertainty relation in SVM is not obvious. Once this is clarified, we may extend it to other systems which contain (sometimes hidden) intrinsically stochastic nature, such as hydrodynamics. The uncertainty relation of hydrodynamics may further offer a clue to find possible quantum effects when hydrodynamic approaches are applied to microscopic systems, like relativistic heavy-ion collisions.

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As is pointed out by Nelson in Ref. [35], when the stochastic trajectory is identified with the real trajectory of a particle, a presumable requirement of physics (separability) cannot be employed. As is seen later, however, the stochastic trajectory does not necessarily have a physical reality in our discussion of the uncertainty relation. Therefore we will not go into the details of the interpretation of the stochastic trajectory in Nelson's stochastic approach.

In this paper, we discuss the generalized uncertainty relation in the framework of SVM. For this purpose, we introduce the Hamiltonian formalism of SVM, which is applicable to particle systems and continuum media of quantum and classical dynamics on an equal footing. This enables us to define the standard deviation of the momentum for general stochastic trajectories and hence to derive the generalized uncertainty relation. As a special case, this reduces to that in quantum mechanics. When it is applied to water at room temperature, we find that the obtained minimum uncertainty is two orders of magnitude larger than that of quantum mechanics, although it is still sufficiently small compared to the coarse-grained scale of hydrodynamics. Such a minimum uncertainty will play an important role in applying hydrodynamics to a microscopic system like relativistic heavy-ion collisions.

This paper is organized as follows. In Sec. 2, we first introduce the Hamiltonian form in SVM. In Sec. 3, we define the standard deviation of the momentum in SVM and derive the inequality between the standard deviations of the position and the momentum satisfied for stochastic systems. The applications to continuum media is discussed in Sec. 4. Section 5 is devoted to concluding remarks.

In the following, k_B and c denote the Boltzmann constant and the speed of light, respectively.

2. Hamiltonian formulation of SVM

2.1. Stochastic Lagrangian

Before introducing the Hamiltonian form, we shortly review the standard formulation of SVM. See for example, Refs. [17,18] for details.

We consider the trajectory of a (virtual) particle described by the following forward stochastic differential equation (SDE),

$$d\mathbf{r}(t) = \mathbf{u}(\mathbf{r}(t), t)dt + \sqrt{2\nu}d\mathbf{W}_t \quad (dt > 0). \tag{1}$$

Here $\mathbf{u}(\mathbf{x}, t)$ is a field associated with the particle velocity yet to be determined by the stochastic variation. In this paper, a difference $dA(t)$ is always defined by $A(t + dt) - A(t)$, independently of the sign of dt. The last term generates the zig-zag nature of the trajectory and called noise term. The intensity of the noise is characterized by ν. We consider that the noise is given by the (standard) Wiener process $\mathbf{W}_t$ which satisfies the following correlation properties,

$$E[d\mathbf{W}_t] = 0, \tag{2}$$

$$E[(dW_t^i)(dW_t^j)] = |dt|\delta^{ij}, \quad (i, j = x, y, z), \tag{3}$$

where $E[\ \ ]$ indicates the average over stochastic events.

The particle motion described by Eq. (1) can be characterized also by introducing the probability distribution defined by

$$\rho(\mathbf{x}, t) = \int d^3\mathbf{r}_i\ \rho_I(\mathbf{r}_i)E[\delta^{(3)}(\mathbf{x} - \mathbf{r}(t))], \tag{4}$$

where $\mathbf{r}(t)$ (more precisely $\mathbf{r}(t; \mathbf{r}_i)$ with $\mathbf{r}_i$ being the initial position of the particle) is the solution of Eq. (1) and $\rho_I(\mathbf{r}_i)$ is the particle distribution at an initial time t_i. As is well-known, the evolution equation of $\rho(\mathbf{x}, t)$ is, using Eq. (1), given by the Fokker–Planck equation,

$$\partial_t\rho(\mathbf{x}, t) = \nabla \cdot (-\mathbf{u}(\mathbf{x}, t) + \nu\nabla)\rho(\mathbf{x}, t). \tag{5}$$

In the formulation of the variational method, we should fix not only an initial condition but also a final condition. This implies that the forward SDE alone is not sufficient. We have to consider also a backward process in time, $dt < 0$, describing a stochastic process from the final condition to the initial condition. That is, when the probability distribution evolves from $\rho_I(\mathbf{x})$ at t_i to $\rho_F(\mathbf{x}) = \rho(\mathbf{x}, t_f)$ at a final time t_f following Eq. (5), the time-reversed process describes the evolution from $\rho_F(\mathbf{x})$ to $\rho_I(\mathbf{x})$. Suppose that this process is given by the backward SDE,

$$d\mathbf{r}(t) = \tilde{\mathbf{u}}(\mathbf{r}(t), t)dt + \sqrt{2\nu}d\tilde{\mathbf{W}}_t \quad (dt < 0), \tag{6}$$

where $\tilde{\mathbf{W}}_t$ is the Wiener process again satisfying the same correlation properties introduced above. These SDEs (1) and (6) are relative to the increasing and decreasing sub-σ-algebras used below to define the mean forward and backward derivatives, respectively [20,29]. For Eq. (6) to describe the same statistical ensemble given by the forward SDE (1), we find that the following consistency condition should be satisfied [18],

$$\mathbf{u}(\mathbf{x}, t) = \tilde{\mathbf{u}}(\mathbf{x}, t) + 2\nu\nabla \ln \rho(\mathbf{x}, t). \tag{7}$$

The same property can be reproduced also from the nature of the Bayesian statistics [37].

For the stochastic trajectories, the usual definition of the particle velocity is not applicable because $d\mathbf{r}/dt$ is not well defined in the vanishing limit of dt due to the singular behavior of $\mathbf{W}_t$ (and $\tilde{\mathbf{W}}_t$). The possible time differential in such a case is studied by Nelson [20], finding that there are two possibilities: one is the mean forward derivative,

$$D\mathbf{r}(t) = \lim_{dt\to 0+} E\left[\frac{\mathbf{r}(t + dt) - \mathbf{r}(t)}{dt}\Big|\mathcal{P}_t\right], \tag{8}$$

and the other the mean backward derivative,

$$\tilde{D}\mathbf{r}(t) = \lim_{dt\to 0-} E\left[\frac{\mathbf{r}(t + dt) - \mathbf{r}(t)}{dt}\Big|\mathcal{F}_t\right]. \tag{9}$$

These expectations are conditional averages, where $\mathcal{P}_t$ ($\mathcal{F}_t$) indicates to fix values of $\mathbf{r}(t')$ for $t' \le t$ ($t' \ge t$). For the σ-algebra of all measurable events of $\mathbf{r}(t)$, $\mathcal{P}_t$ and $\mathcal{F}_t$ represent an increasing and a decreasing family of sub-σ-algebras [12]. Then, applying these definitions to Eqs. (1) and (6), we obtain $D\mathbf{r}(t) = \mathbf{u}(\mathbf{r}(t), t)$ and $\tilde{D}\mathbf{r}(t) = \tilde{\mathbf{u}}(\mathbf{r}(t), t)$, respectively.

To see how we introduce actions expressed in the above stochastic trajectory $\mathbf{r}(t)$, let us consider the classical Lagrangian for one particle, $L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{m}{2}\dot{\mathbf{r}}^2(t) - V(\mathbf{r}(t))$ where m is the mass of the particle and $V(\mathbf{x})$ is a potential distributed in $\mathbf{x}$. Due to the existence of the two definitions of the time derivatives D and $\tilde{D}$, the most general quadratic form of the kinetic energy in terms of the stochastic trajectory is given by [19]

$$\frac{m}{2}\dot{\mathbf{r}}^2(t) \longrightarrow \frac{m}{2}\left[B_+\{A_+(D\mathbf{r}(t))^2 + A_-(\tilde{D}\mathbf{r}(t))^2\} + B_-(D\mathbf{r}(t)) \cdot (\tilde{D}\mathbf{r}(t))\right], \tag{10}$$

where

$$A_\pm = 1/2 \pm \alpha_1 \quad B_\pm = 1/2 \pm \alpha_2, \tag{11}$$

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