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Novel algebraic aspects of Liouvillian integrability for two-dimensional polynomial dynamical systems

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ABSTRACT

The general structure of irreducible invariant algebraic curves for a polynomial dynamical system in $\mathbb{C}^2$ is found. Necessary conditions for existence of exponential factors related to an invariant algebraic curve are derived. As a consequence, all the cases when the classical force-free Duffing and Duffing–van der Pol oscillators possess Liouvillian first integrals are obtained. New exact solutions for the force-free Duffing–van der Pol system are constructed.

1. Introduction

Integrating a dynamical system is one of the major problems of analysis. Existence of invariant algebraic curves and exponential factors is a substantial measure of integrability. In this article our goal is to derive the general structure of irreducible invariant algebraic curves for a generic two-dimensional polynomial dynamical system. We apply our results to the famous Duffing and Duffing–van der Pol oscillators arising in a variety of applications, see [1] and references therein. These models and their generalizations have been intensively studied in recent years [2–7]. We solve completely the problem of Liouvillian integrability for the classical force-free Duffing and Duffing–van der Pol oscillators. It seems that this problem for generic values of the parameters has not yet been studied.

A polynomial vector field in $\mathbb{C}^2$ can be defined as

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}, \quad P(x, y), Q(x, y) \in \mathbb{C}[x, y]. \quad (1.1)$$

By $\mathbb{C}[x, y]$ we denote the ring of polynomials in the variables x and y with coefficients in $\mathbb{C}$. The dynamical system associated to X reads as

$$x_t = P(x, y), \quad y_t = Q(x, y). \quad (1.2)$$

A non-constant function I: $\mathbb{C}^2 \to \mathbb{C}$ is called a first integral of the polynomial vector field X on an open subset $D \subset \mathbb{C}^2$ if $I(x(t), y(t)) = C$ with C being a constant for all values of t such that the solution $(x(t), y(t))$ of X is defined on D. We say that the vector field X is Liouvillian integrable if there exists a Liouvillian first integral I of X. Generally speaking, a function is Liouvillian if it can be expressed using quadratures of elementary functions, for more details and strict definitions see [8–10].

An algebraic curve $F(x, y) = 0$, $F(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}$ is an invariant algebraic curve (or a Darboux polynomial) of the vector field X if it satisfies the following equation $XF = \lambda(x, y)F$, i.e.

$$P(x, y)F_x + Q(x, y)F_y = \lambda(x, y)F, \quad (1.3)$$

where $\lambda(x, y) \in \mathbb{C}[x, y]$ is a polynomial called the cofactor of the invariant curve $F(x, y)$. It is straightforward to find that the polynomial $\lambda(x, y)$ is of degree at most $m - 1$, where m is the degree of X: $m = \max\{\deg P, \deg Q\}$, [8,9].

The function $E = \exp(g/f) \notin \mathbb{C}$ with coprime polynomials $g, h \in \mathbb{C}[x, y]$ is an exponential factor of the vector field X whenever it satisfies the equation $XE = \varrho(x, y)E$. The polynomial $\varrho(x, y) \in \mathbb{C}[x, y]$ is called the cofactor of the exponential factor E and is of degree at most $m - 1$. It can be easily shown that if the exponential factor $E = \exp(g/f)$ contains a non-constant polynomial f, then f is an invariant algebraic curve of X [11,12]. The exponential factors arise, when invariant algebraic curves degenerate [11,12]. This concept of degenerate invariant algebraic curves was introduced by Christopher [11,12].

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A non-constant function $R: \mathbb{C}^2 \to \mathbb{C}$ is an integrating factor of the polynomial vector field X on an open subset $D \subset \mathbb{C}^2$ if $XR = -R\,\mathrm{div}(X)$ on D. Recall that $\mathrm{div}(X) = P_x + Q_y$.

It is known that the problem of proving Liouvillian integrability or non-integrability of a polynomial vector field X and associated dynamical system (1.2) can be reduced to the problem of constructing all irreducible invariant algebraic curves of X and all exponential factors of X [8–12]. Note that there exist Liouvillian integrable polynomial dynamical systems that have no invariant algebraic curves [13].

The following powerful theorems are valid.

Theorem 1.1. *Let $F_j = 0$, $j = 1, \dots, r \in \mathbb{N}_0$ be irreducible invariant algebraic curves of X and $E_k = \exp(g_k/f_k)$, $k = 1, \dots, s \in \mathbb{N}_0$ be exponential factors of X. Then the function*

$$F_1^{d_1} \dots F_r^{d_r} E_1^{e_1} \dots E_s^{e_s}, \quad d_1, \dots, d_r \in \mathbb{C}, \quad e_1, \dots, e_s \in \mathbb{C}, \tag{1.4}$$

is a first integral (possibly, multi-valued) of X if and only if d_j, e_k are not all zero and the following condition is valid

$$\sum_{j=1}^{r} d_j \lambda_j + \sum_{k=1}^{s} e_k \varrho_k = 0, \tag{1.5}$$

where λ_j is a cofactor of F_j, ϱ_k is a cofactor of E_k.

Note that the function of the form (1.4) is called a Darboux function. Obviously, any Darboux function is Liouvillian.

Theorem 1.2. *Under the assumptions of Theorem 1.1 the polynomial vector field X has a Liouvillian first integral if and only if it has an integrating factor of the form (1.4).*

Theorem 1.3. *Under the assumptions of Theorem 1.1 the Darboux function given in (1.4) is an integrating factor of X if and only if the following condition is valid*

$$\sum_{j=1}^{r} d_j \lambda_j + \sum_{k=1}^{s} e_k \varrho_k = -div(X). \tag{1.6}$$

Theorems 1.1 and 1.3 follow from classical theory of Darboux integrability, see [8,9,14]. Theorem 1.2 was proved by Singer [10] and further strengthened by Christopher [11]. A considerable impact in the study of Liouvillian integrability of polynomial dynamical systems has been made by Llibre and Valls, see for example [15,16].

Our aim in the present article is to establish the structure of irreducible invariant algebraic curves of a polynomial vector field X. Regarding the variable y as dependent and the variable x as independent, we find that the function $y(x)$ satisfies the following first-order ordinary differential equation

$$P(x, y) y_x - Q(x, y) = 0. \tag{1.7}$$

In what follows we suppose that the polynomials $P(x, y)$, $Q(x, y)$ do not have non-constant common factors.

A Puiseux series in a neighborhood of the point $x = \infty$ is defined as

$$y(x) = \sum_{k=0}^{+\infty} b_k x^{\frac{l_0}{n} - \frac{k}{n}} \tag{1.8}$$

where $l_0 \in \mathbb{Z}$, $n \in \mathbb{N}$. It follows from the classical results that a Puiseux series of the form (1.8) that satisfy the equation $F(x, y) = 0$, $F(x, y) \in \mathbb{C}[x, y]$ is convergent in a neighborhood of the point $x = \infty$ (the point $x = \infty$ is excluded from domain of convergence if $l_0 < 0$) [17]. The set of all Puiseux series of the form (1.8) forms a field, which we denote by $\mathbb{C}_\infty\{x\}$. In the next section we shall prove that if $y(x)$ is a Puiseux series solving the equation $F(x, y) = 0$, $F_y \not\equiv 0$ with $F(x, y)$ being an invariant algebraic curve of the polynomial vector field X, then the series $y(x)$ satisfies equation (1.7).

Recall that if $n > 1$ then the convergence of the corresponding series is understood in the sense that a certain branch of the n-th root is chosen and a cut forbidding going around the branch point is introduced. Further, there exists a compact subset in the domain of convergence of the Puiseux series satisfying the equation $F(x, y) = 0$ such that convergence of the corresponding series is uniform.

All the Puiseux series that solve equation (1.7) can be obtained with the help of the Painlevé methods [14,18–20], for more details see section 3.

The main contributions of the present article are the following five theorems.

Theorem 1.4. *Let $F(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}$, $F_y \not\equiv 0$ be an irreducible invariant algebraic curve of polynomial vector field X and related dynamical system (1.2). Then $F(x, y)$ takes the form*

$$F(x, y) = \left\{ \mu(x) \prod_{j=1}^{N} \left\{ y - y_j(x) \right\} \right\}_+, \quad N \in \mathbb{N}, \tag{1.9}$$

where $\mu(x) \in \mathbb{C}[x]$ and $y_1(x), \dots, y_N(x)$ are pairwise distinct Puiseux series in a neighborhood of the point $x = \infty$ that satisfy equation (1.7). The symbol $\{W(x, y)\}_+$ means that we take the polynomial part of the expression $W(x, y)$. Moreover, the degree of $F(x, y)$ with respect to y does not exceed the number of distinct Puiseux series of the form (1.8) satisfying equation (1.7) whenever the latter is finite.

Consequence. If neither Puiseux series in a neighborhood of the point $x = \infty$ satisfy equation (1.7) nor the functions of the form (1.9) constructed with the help of all possible combinations of admissible Puiseux series are polynomial, then invariant algebraic curves of vector X (if any) are of the form $F(x)$. Here an admissible Puiseux series means that this series is of the form (1.8) and satisfies equation (1.7).

The inverse theorem is also valid.

Theorem 1.5. *Suppose that $y_1(x), \dots, y_N(x)$ are pairwise distinct Puiseux series in a neighborhood of the point $x = \infty$ that satisfy equation (1.7). Let the polynomial $\mu(x) \in \mathbb{C}[x]$ be such that the following expression*

$$F(x, y) = \mu(x) \prod_{j=1}^{N} \left\{ y - y_j(x) \right\} \tag{1.10}$$

is an irreducible in $\mathbb{C}[x, y]$ polynomial, i.e. the non-polynomial part in (1.10) vanishes producing the polynomial $F(x, y)$, then $F(x, y)$ is an irreducible invariant algebraic curve of the polynomial vector field X and related dynamical system (1.2).

Remark. Theorems 1.4 and 1.5 introduce a novel algebraic tool for finding all irreducible invariant algebraic curves explicitly. At the first step, one should obtain all the Puiseux series near the point $x = \infty$ that satisfy equation (1.7). At the second step, it is necessary to consider different combinations of the Puiseux series and require that the non-polynomial part of expression (1.10) vanishes. Restrictions on the polynomial $\mu(x)$ (if any) can be found balancing higher-order terms (with respect to y) in equation (1.3).

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